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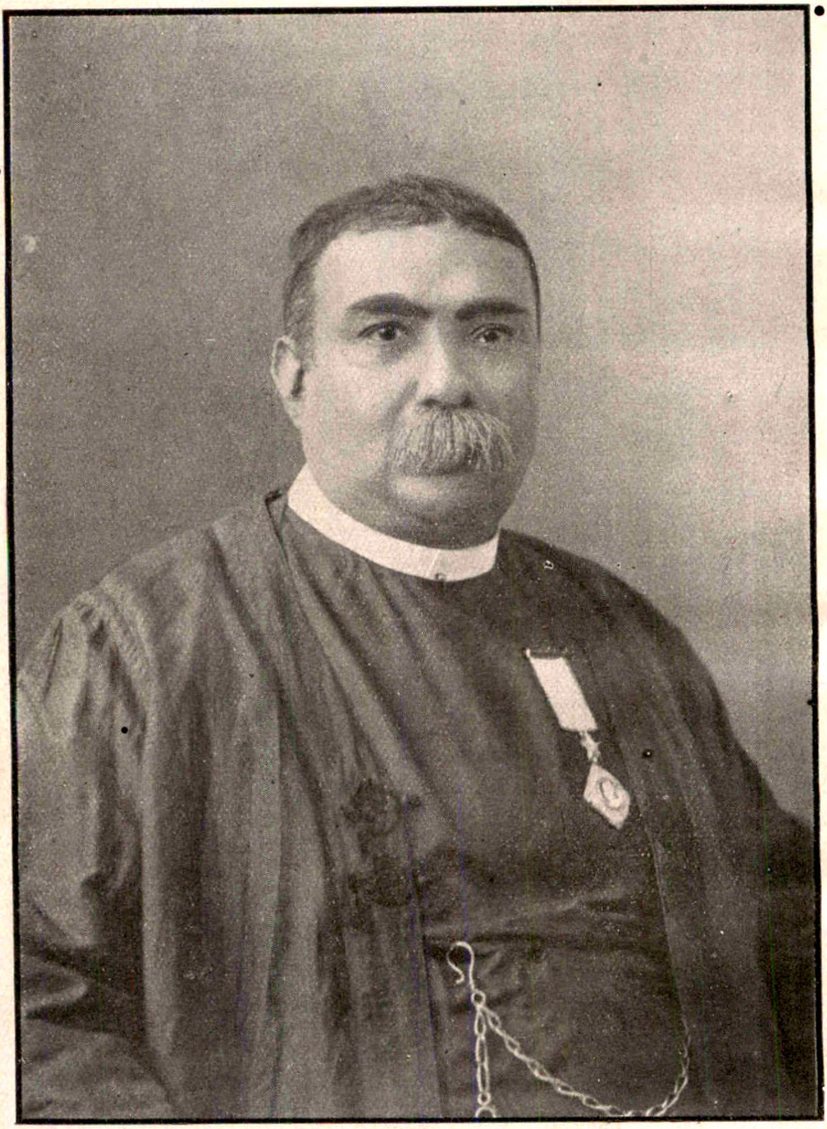

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1

Sur la distance de deux ensembles

PAR

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Introduction.—Dans une note "Sur les ensembles mesurables" publiée dans les Comptes Rendus de l'Académie des Sciences, 1923, page 69, M. Tadé Wazewski obtient un certain nombre de propriétés des suites d'ensembles mesurables en appliquant une définition de la distance de deux ensembles mesurables qu'il attribue à M. Nikodym.

Je me propose de montrer ici comment les résultat de M. Wazewski et de M. Nikodym peuvent être considérés comme cas particuliers de propositions que j'ai obtenu précédemment concernant les suites de fonctions mesurables. Il suffira pour cela de supposer qu'on limite ces propositions au cas où les fonctions qu'on considére ne peuvent prendre que les valeurs zéro et un. Alors chacune de ces fonctions pourra être considérée comme la "fonction caractéristique" d'un ensemble—mesurable en même temps que la fonction—à savoir la fonction égale à un sur cet ensemble et nulle en dehors.

Distance de deux ensembles.—Dans certaines théories, l'influence d'un ensemble est pour ainsi dire massique ; il importe peu qu'on enlève ou qu'on ajoute un point à cet ensemble ; il importe même peu qu'on lui enlève ou qu'on lui ajoute une infinité de points pourvu que tous ceux-ci restent compris dans un ensemble de mesure nulle. C'est en vue de l'utilisation de nos résultats dans ces théories que nous allons nous placer.

Nous avions convenu de considérer deux fonctions comme non distinctes lorsqu'elles ne différent que dans un ensemble de mesure nulle. Si ces deux fonctions sont les fonctions caractéristiques de deux ensembles linéaires E, F, cela revient à dire que l'ensemble des points de E qui n'appartiennent pas à F et des points de F qui n'appartiennent pas à E est de mesure nulle.

Ainsi on ne considérera pas comme distincts deux ensembles E, F tels que

$$\text{mesure de } [(E-F)+(F-E)]=0.$$

II Nous avions défini[1] la distance de deux fonctions $f(x)$, $\phi(x)$ comme la borne inférieure (f, ϕ) de la somme

$$w+m_w$$

quand w varie en restant positif, en désignant par m_w la mesure *extérieure* au sens de M. Lebesgue de l'ensemble des points où $|f(x) -\phi(x)|>w$. Si f,ϕ sont les fonctions caractéristiques de deux ensembles E, F on voit que si $w \geq 1$, $m_w=0$, donc $w+m_w$ reste supérieur ou égal à un et si $1>w>0$, m_w est la mesure extérieure μ de l'ensemble $(E-F)+(F-E)$ de sorte que $w+m_w=w+\mu$ est aussi voisin que l'on veut de μ en prenant w assez petit. Si l'on suppose [2] que l'intervalle de définition des fonctions est l'intervalle (0,1), on voit que la borne inférieure de $w+m_w$ est égal à μ. Finalement si on convient d'appeler *distance de* E à F et de représenter par (E, F) la distance de leurs fonctions caractéristiques, on aura

$$(E, F)=\text{mesure extérieure de } [(E-F)+(F-E)]$$

Il résulte alors des proprietés démontrées[2] pour la distance (f, ϕ) que la distance de deux ensembles jouit des propriétés suivants.

I La condition nécessaire et suffisante pour que deux ensembles linéaires E, F ne différent chacun de leur ensemble commun que par un ensemble de mesure nulle est que leur distance (E, F) soit nulle:

II On a $(E, F)=(F, E)$. Et si l'on substitue à E, F deux ensembles E_1, F_1 équivalents au sens précédent à E, F, on a

$$(E_1, F_1)=(E, F)$$

III Quels que soient les ensembles linéaires E, F, G, on a

$$(E, F)\leq(E, G)+(G, F)$$

Seulement on remarquera que si notre definition coïncide avec celle de M. Nikodym dans le cas, visé par celui-ci, où les ensembles linéaires considérés sont mesurables, elle subsiste et vérifie encore les conditions I, II, III pour des ensembles linéaires quelconques.

Elément-limite.—Dans le mémoire cité plus haut, nous établissions que la classe des fonctions quelconques d'une variable numérique x est une classe (D), c'est à dire qu'on peut y définir une distance compatible avec la définition de la convergence des suites de fonctions si l'on a choisi pour celle-ci la convergence *en mesure* de F. Riesz—généralisée au cas des fonctions quelconques.

[1] Bull, Calcutta Math. Soc., Vol. XI, 1921.

[2] Dans le cas contraire, on prendrait pour fonction caractéristique d'un ensemble une fonction égale sur l'ensemble à la longueur de l'intervalle.

Nous avons donc en somme prouvé une propriété importante d'une définition de la convergence introduite précédemment et qui a rendu des services en Analyse.

Si au contraire, on introduit a priori comme M. Nikodym une définition de la distance de deux fonctions mesurables, on ne prouve pas par là qu'une telle definition est possible ; cela est possible dans toute classe et d'une infinité de manières ; ce qui n'est pas toujours possible c'est de definir une distance compatible avec une dèfinition de la limite donnée. Mais inversement on introduit par la même une définition de la convergence d'une suite et il est utile de voir en qui consiste cette convergence, ou ce qui sont ses propriétes.

D'après la façon dont nous avons opéré on voit que si l'on convient qu'une suite d'ensembles linéaires E_1, E_2,...converge vers un ensemble E lorsque la distance (E, E_n) tend vers zéro, on retombe sur un cas particulier de la convergence en mesure de F. Riesz. Comment se traduit-il lorsque les fonctions considérées par F. Riesz sont des fonctions caractéristiques d'ensemble ?

La condition nécessaire et suffisante pour que la distance (E, E_n) des deux ensembles linéaires E et E_n, converge vers zéro avec $\frac{1}{n}$ est que les points de E et de E_n qui ne sont pas communs à ces deux ensembles puissent être enfermés dans un ensemble dénombrable I_n d'intervalles dont la longueur totale tend vers zéro avec $\frac{1}{n}$.

Il y a lieu de faire remarquer que l'ensemble I_n d'intervalles est variable avec n et en outre que si la mesure de I_n tend vers zéro, l'ensemble commun aux I_n n'est pas nécessairement de mesure nulle. Enfin les ensembles limite complet et restreint, C et R, des ensembles E_n, peuvent aussi différer sur un ensemble de mesure non nulle. L'example donné dans le mémoire précédemment cité relalif au cas des fonctions s'adapte immédiatement au cas des ensembles. Divisons l'intervalle fixe J où sont situés les ensembles considérés en 2 intervalles égaux J_1, J_2, puis en 4 intervalles égaux J_3, J_4, J_5, J_6, puis en huit, et.

Prenons alors pour ensemble E_n l'intervalle J_n ; il est manifeste que la suite des distances des J_n à l'ensemble E vide de tout point tend vers zéro. Pourtout l'ensemble limite complet des J_n est constitué par l'intervalle fondamental J tout entier, alors que leur ensemble limite restreint est vide comme E.

Il est par contre toujours possible d'extraire d'une suite d'ensembles E_n qui convergent en mésure vers un ensemble E une suite particuliére E_{n_1}, E_{n_2}, pour laquelle non seulement les ensembles limites complet et resteint de cette suite particuliére coïncident à un ensemble de

mesure nulle près, mais encore telle que les ensembles d'exclusion I_{n_1}, I_{n_2},...soient chacun compris dans le précédent.

En d'autres termes il est possible de choisir parmi les E_n, une suite particulière E_{n_1}, E_{n_2}, telle que quel que soit E il existe une ensemble d'intervalles de longueur tolale inférieure à E et en dehors desquels les E_{n_p} sont identiques (et identiques à E) à partir d'un certain rang.

Cette proposition s'obtient immediatement en appliquant au cas des fonctions caractéristiques un théorème concernant les suites de fonctions convergeant en mesure, théorème démontré d'abord par F. Riesz pour les fonctions mesurables et ètendu dans mon mèmoire aux fonctions quelconques d'une variable.

Il résulte en particulier de ce qui précéde que les ensembles limites complet et restreint C' et R' de la suite des E_{n_p} sont identiques à un ensemble près de mesure nulle ce qui constitue la proposition IV de M. Wazewski. Mais la proposition actuelle est plus precise.

Enfin la proposition II de M. Wazewski s'obtient immediatement si l'on applique aux fonctions caractéristiques d'ensembles, le résultat démontré dans mon mémoire concernant la condition de convergence en mesure d'une suite de fonctions. On peut alors dire que :

Pour qu'une suite d'ensemble, E_1, E_2,...converge en mesure il faut et il suffit que quel que soit E la distance (E_n, E_{n+p}) de l'un E_n de ces ensembles à l'un quelconque E_{n+p} des suivants, soit inférieure à E quel que soit p, pour n assez grand.

Autrement dit, quand on definit la convergence d'une suite d'ensembles *par la convergence en mesure, la classe des ensembles linéaires est une classe* (D) *complète.*

.——D'autre part, j'ai montré ailleurs quelle importance ont pour une classe (D) certaines propriétés générales qui permettent d'étendre à cette classe un grand nombre de théorèmes de la théorie des ensembles linéaires. Etablissons ici certaines de ces propriétés pour la classe des ensembles linéaires.

.——D'abord cette classe est évidemment parfaite. Autrement dit quel que soit l'ensemble linéaire E il existe un ensemble F distinct de E au sens adopté plus haut et dont la distance à E est aussi petite que l'on veut. Il suffit pour former F d' ajouter à E ou de supprimer de E, suivant le cas, un intervalle de longueur aussi petite qu'on voudra.

.——On peut aussi joindre deux éléments d'un même sphéroïde par un arc de Jordan contenu dans ce sphéroïde.

Autrement dit si l'on considére deux ensembles linéaires E, F dont les distances à un ensemble C sont au plus égales à ρ, il est possible de definir

un ensemble linéaire G_t dependant d'un paramètre t de sorte que $G_0=E$, $G_1=F$, que $(C, G_t) \leq \rho$ pour $0 \leq t \leq 1$ et enfin tel que $(G_t, G_{t'})$ tende vers zéro quand $(t'-t)$ tend vers zéro.

Il suffit évidemment de montrer que cela est possible quand l'un des eléments E, F;—F par example—est au centre du sphéroïde. Car alors il suffira dans le cas général de joindre E à C et C à F.

Or considérons l'ensemble

$$U=(E-C)+(C-E)$$

et l'ensemble U_t des points de $E-C$ qui sont situés dans l'intervalle o, t, et de $C-E$ qui sont situés dans l'intervalle $(t, 1)$. On a :

$U_0=C-E$ et $U^1=E-C$; donc en posant $G_t=C.E+U_t$ on aura

$$G_0=C \text{ et } G_1=E$$

D'autre part l'ensemble $(C-G_t)+(G_t-C)$ se compose de la partie de $(C-E)$ comprise dans (o, t) et de la partie de $E-C$ comprise de o à t. Donc la distance (C, G_t) croit constamment ou du moins ne décroit pas quand t croit ; ses valeurs extrêmes sont o et $(C, G_1) =(C, E) \leq \rho$. Finalement G_t reste bien compris dans le sphéroïde de centre C. Enfin, $(G_t-G_{t'})+(G_{t'}-G_{t'})$ est un ensemble compris dans l'intervalle (t, t'), par conséquent

$$(G_t, G_{t'}) \leq (t'-t)$$

et l'elément G_t dépend continuement de t.

.——Donc ce qui précède s'applique à des ensembles linéaires quelconques. Nous voyous ainsi que lorsqu'on adopte pour définition de la convergence d'une suite d'elément la convergence en mesure, la classe des ensembles linéaires est une classe (D) parfaite, complète et où deux eléments quelconques d'un sphéroïde peuvent être joints par un arc de Jordan appartenant à ce sphéroïde. Cette classe possède donc toutes les propriétés que j'ai enoncées concernant les classes de cette espèce dans mon mémoire "Esquisse d'une théorie des ensembles abstraits, University of Calcutta, 1922."

.——*Cas des ensembles mesurables.*—Il en est de même dans le cas où on restreint la classe aux ensembles linéaires mesurables. Il suffit en effet de remarquer que dans les démonstrations précédentes, si on suppose que les ensembles donnés sont mesurables, les ensembles construits à partir de ceux-ci et utilisés dans ces démonstrations sont aussi mesurables.

Mais on peut énoncer en outre une propriété spéciale à la classe des ensembles linéaires mesurables, à savoir que *cette classe est séparable.* Nous avons indiqué en effet dans le second mémoire cité (page 389)

que la classe des fonctions mesurables peut être considérée comme l'ensemble dérivé d'un de ses ensembles dénombrables, à savoir l'ensemble des fonctions qui sont constantes et de valeurs rationnelles dans chacune des sub-divisions de l'intervalle fondamental limitées par un nombre fini variable de points d'abscisses rationnelles. Dans le cas où on prend comme fonction des fonctions caractéristiques d'ensembles, on voit que la classe des ensembles mesurables peut être considérée comme l'ensemble dérivé de l'ensemble dénombrable N dont chaque élément est l'ensemble linéaire mesurable constitué par la reunion d'un nombre fini d'intervalles à extrémités rationnelles. C'est en utilisant également l'ensemble dénombrable N que la même proposition a été établie par M. Wazewski.

.——Il est alors loisible d'appliquer à la classe des ensembles mesurables le même théorème que j'ai démontré pour les fonctions mesurables. Tout ensemble (d'ensembles linéaires mesurables) est condensé. En effet nous savons que cela est vrai pour tout ensemble tiré d'une classe (D) séparable.

Je rappelle la signification du théorème obtenu. Etant donné un ensemble quelconque F, d'ensembles linéaires mesurables, tout sous-ensemble non dénombrable G de F donne lieu à au moins un elément de condensation E. C'est à dire qu' il existe un ensemble linéaire mesurable E qui est élément limite d'une suite convergeant en mesure d'ensembles tirés de G et aussi d'une suite convergeant en mesure d'ensemble tirés de G—N quel que soit le sous-ensemble dénombrable N de G.

.——Il en résulte en particulier ce corollaire que : de tout ensemble non dénombrable d'ensemble linéaires mesurables on peut tirer une suite convergeant en mesure ; et comme de celle-ci on peut extraire une suite d'ensembles satisfaisant à la condition plus précise indiquée plus haut, on peut dire en résumé.

De tout ensemble non dénombrable F d'ensembles linéaires mesurables, on peut tirer une suite d'ensembles $F_1, F_2 \ldots F_n \ldots$ qui coïncident entre eux à partir de chaque rang n en dehors d'un ensemble I_n composé d'une suite dénombrable d'intervalles dont la longueur totale tend vers zéro avec $\frac{1}{n}$, I_{n+1}, étant compris dans I_n.

Il en résulte en particulier que l'ensemble limite complet et l'ensemble limite restreint des F_n coïncident en dehors d'un ensemble de mesure nulle (l'ensemble commun aux I_n)—ce qui donne la proposition I de M. Wazewski moins précise que la précédente.

.——*Examen de diverses definitions de la convergence.*—Dans ce qui précède nous avons admis qu'on prenait comme définition de la convergence d'une suite d'eléments, la convergence en mesure, l'ensemble

linéaire E_n convergeant en mesure vers E si E_n et E coïncident en dehors d'un ensemble dénombrable I_n d'intervalles dont la longueur totale tend vers zéro avec $\frac{1}{n}$. Nous avons vu que dans ces conditions, on peut definir la convergence par l'intermédiaire d'une distance.

D'autres definitions de la convergence paraitraient plus naturelles et il est important de montrer qu'elles ont l'inconvénient de ne pas reprêter à l'intermédiaire d'une distance, en même d'un écart.

Par exemple on pourrait dire que E_n converge si ses ensembles limites complet et restreint coincident. Mais si l'on adoptait une telle definition et si elle pourrait se traduire par l'intermédiaire d'une distance tout ensemble dérivé d'un ensemble d'ensembles linéaires serait fermé. Or nous savons qu'il n'en est pas ainsi ; car en partant par exemple de l'ensemble E des ensembles linéaires formés d'un nombre fini d'intervalles, on trouve comme ensembles derivé E^1, un ensemble d'ensembles linéaires qui n'est pas fermé !

On ne pourrait même pas traduire la definition actuelle de la convergence par l'intermédiaire d'un "écart." Autrement dit, quel que soit la façon dont on ferait correspondre à tout couple d'ensembles linéaires E, F un nombre [E, F] qu'on appellerait écart de E, F, il serait impossible de satisfaire aux conditions suivantes :

I $[E, F]=0$ est la condition nécessaire et suffisante pour que E, F coïncident.

II Quels que soient E, F, on a $[E, F]=[F, E]\geq 0$.

III La condition nécessaire et suffisante pour que $[E, E_n]$ tende vers zéro est que les ensembles limites complet et restreint de E_n, coïncident avec E.

En effet, désignons par V_q l'ensemble de point communs à tous les ensembles F tels que $[E, F] <\frac{1}{q}$. Parmi ces ensembles F figure E lui même, donc V_q est compris dans E, quelque soit q. Ainsi E comprend $V_1+V_2+\ldots$; ces deux ensembles sont même identiques ; car si un point x de E n'appartenait pas à $V_1+V_2+\ldots$ il n'appartiendrait pas par exemple à V_q. Si donc on considére une suite E_1, E_2,...convergeant vers E, comme $[E, E_n]$ tend vers zéro on aurait $[E, E_n]<\frac{1}{q}$ pour n assez grand par exemple $n>p$. Donc les E_n comprendrait V_q à partir du rang $n=p+1$, et par suite x n'étant compris dans aucun des ensembles E_{p+1}, E_{p+2},...ne pourrait être compris dans leur ensemble-limite E, contrairement à l'hypothèse. Ainsi.

$$E=V_1+V_2+\ldots$$

Ceci étant prenons pour E un ensemble non dénombrable. Alors V_1, V_2,...ne pourront ne contenir chacun qu'un nombre fini de points. On

pourra donc choisir une infinité de points distincts $x_1, x_2,\ldots x_n\ldots$ dans l'un au moins de ces ensembles par exemple dans V_r.

Considerons alors les ensembles $F_n = E - x_n$; ils ont évidemment E pour ensemble limite complet et restreint. Donc pour n assez grand, par exemple pour $n \geq m$, on aura $[E, E_n] < \frac{1}{r}$, alors V_r sera compris dans E_m, $E_{m+1},\ldots$et on arrive à une contradiction puisque x_m est compris dans V_r sans l'être dans E_m.

Ainsi l'hypothèse de l'existence d'un écart doit être écartée :

Convergent presque uniforme.—Nous avons suivi ici pour le cas des ensembles la méthode qui nous a réussi dans notre premier mémoire de Calcutta concernant la convergence ordinaire des fonctions. Operons de même pour la convergence presque uniforme.

Celle-ci semble plus naturelle et plus utile que la convergence en mesure et pourtant on la rencontre plus exceptionnellement. Elle consiste en ceci : une suite d'ensembles linéaires E_1, $E_2,\ldots$converge vers E si quel que soit E, on peut assigner un ensemble dénombrable d'intervalles de longueur totale inférieure à E tel qu'en dehors de cet ensemble d'intervalles les ensembles E_n coïncident avec E à partir d'un certain rang.

Montrons qu'on arrive à une contradiction si l'on essaie de definir cette convergence presque uniforme au moyen d'un écart.

Divisons comme précédemment l'intervalle fondamental (0, 1) en 2 puis 4 puis 8,...puis 2^n parties égales, et appelons I_n^1, $I_n^2,\ldots I_n^s,\ldots$ces 2^n parties.

Il est manifeste comme nous l'avons fait déja remarquer que la suite I_2^1, I_2^2, I_4^1, I_4^2, I_4^3, $I_4^4,\ldots I_m^n$, I_m^{n+1}, ne converge uniformément vers aucun ensemble. En particulier elle ne converge pas uniformément vers un ensemble vide et par suite l'écart $[I_m^s, 0]$ ne tend pas vers zéro. Si l'on appelle k_m le plus grand de ces écarts pour m fixe, k_m ne tend donc pas vers zéro. Autrement dit, il existe un nombre $\lambda > 0$ tel qu'une infinité des k_m, soit k_{m_1}, k_{m_2}, k_{m_p}, restent supérieurs à λ. Soit alors J_p celui des intervalles $I_{m_p}^s$ dont l'écart avec zéro est k_{m_p} et C_p son centre. On peut extraire de la suite des C_p une suite convergente, C'_1, $C'_2,\ldots C'_r,\ldots$; soit C son point limite ; appelons J'_1, $J'_2,\ldots$les intervalles J_p correspondant à C'_1, $C_2,\ldots$. Quel que soit le nombre $\epsilon > 0$, on peut toujours prendre p assez grand $(p > r)$ pour que C'_p et même J'_p soient situés dans un intervalle de centre C de longueur ϵ. Par conséquent la suite des J'_p converge presque uniformément vers un ensemble nul. Donc $[J'_p, 0]$ devait tendre vers zéro. Or nous savons que $[J'_p, 0] > \lambda > 0$.

2

Osculating Conics for the Curve $f(x,y)=0$

By

Gurudas Bhar, M.Sc.

An exhaustive treatment of equations of osculating conics of an algebraic curve $x=F(t)$ and $y=G(t)$ has been given by Prof. S. Mukhopadhyaya.[1] I have here attempted to extend the results for an algebraic curve given in the form $f(x,y)=0$.

1

Let $f(x,y)=0$ be the equation of an algebraic curve where $f(x,y)$ may be supposed to be a rational integral function in x and y. Also let the suffixes 1 and 2 denote partial defferentiation with respect to x and y respectively, *i.e.*, let $f_1, f_2, f_{11}, f_{12}, f_{22}$, etc., stand respectively for

$$\frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ \frac{\partial^2 f}{\partial x^2},\ \frac{\partial^2 f}{\partial x \partial y},\ \frac{\partial^2 f}{\partial y^2},\ \text{etc.}$$

If as usual p, q, r, s denote the successive differential co-efficients of y with respect to x, we have

$$p=\frac{dy}{dx}=-\frac{f_1}{f_2},$$

$$q=\frac{d^2y}{dx^2}=\frac{1}{f_2^3}\begin{vmatrix} f_{11} & f_{12} & f_1 \\ f_{21} & f_{22} & f_2 \\ f_1 & f_2 & 0 \end{vmatrix}=\frac{J}{f_2^3},$$

[1] S. Mukhopadhyaya—A General Theory of Osculating Conics—*Journal and Proceedings, Asiatic Society of Bengal* (New Series), Vol. IV, No. 4, 1908; Vol. IV, No. 10, 1908.

where
$$J \equiv \begin{vmatrix} f_{11} & f_{12} & f_1 \\ f_{21} & f_{22} & f_2 \\ f_1 & f_2 & 0 \end{vmatrix},$$

$$r = \frac{d^3y}{dx^3} = -\frac{1}{f_2^{\,5}} \begin{vmatrix} J_1 & J_2 & 3J \\ f_{21} & f_{22} & f_2 \\ f_1 & f_2 & 0 \end{vmatrix} = -\frac{K}{f_2^{\,5}},$$

where
$$K \equiv \begin{vmatrix} J_1 & J_2 & 3J \\ f_{21} & f_{22} & f_2 \\ f_1 & f_2 & 0 \end{vmatrix} \text{ and } J_1 = \frac{\partial J}{\partial x},\quad J_2 = \frac{\partial J}{\partial y};$$

and
$$s = \frac{d^4y}{dx^4} = \frac{1}{f_2^{\,7}} \begin{vmatrix} K_1 & K_2 & 5K \\ f_{21} & f_{22} & f_2 \\ f_1 & f_2 & 0 \end{vmatrix}, \text{ where } K_1 = \frac{\partial K}{\partial x},\ K_2 = \frac{\partial K}{\partial y}.$$

Let us further write

$$P \equiv \begin{vmatrix} J_1 & J_2 & 3J & 0 \\ f_{11} & f_{12} & f_1 & f_1 \\ f_{21} & f_{22} & f_2 & 2f_1\cos\omega - f_2 \\ f_1 & f_2 & 0 & 0 \end{vmatrix},$$

$$Q \equiv \begin{vmatrix} J_1 & J_2 & 3J & 0 \\ f_{11} & f_{12} & f_1 & 2f_2\cos\omega - f_1 \\ f_{21} & f_{22} & f_2 & f_2 \\ f_1 & f_2 & 0 & 0 \end{vmatrix},$$

$$R \equiv \begin{vmatrix} J_1 & J_2 & 3J & 0 \\ f_{11} & f_{12} & f_1 & f_2 \\ f_{21} & f_{22} & f_2 & f_1 \\ f_1 & f_2 & 0 & 0 \end{vmatrix},$$

$$S \equiv \begin{vmatrix} 1 & \cos\omega & f_1 \\ \cos\omega & 1 & f_2 \\ f_1 & f_2 & 0 \end{vmatrix},$$

$$M \equiv \begin{vmatrix} J_1 & J_2 & 3J \\ f_{21} & f_{22} & f_2 \\ f_1 & f_2 & 0 \end{vmatrix}, \quad N \equiv \begin{vmatrix} J_1 & J_2 & 3J \\ f_{11} & f_{12} & f_1 \\ f_1 & f_2 & 0 \end{vmatrix},$$

and

$$T \equiv \begin{vmatrix} 1 & \cos\omega & M \\ \cos\omega & 1 & N \\ M & N & 0 \end{vmatrix}.$$

2

The general equation in oblique co-ordinates of a conic passing through two given points (x,y) and (x_1,y_1) is of the form

$$\lambda(X-x)(X-x_1)+\mu(Y-y)(Y-y_1)+\nu(X-x)(Y-y_1)$$
$$+\rho(X-x_1)(Y-y)=0.$$

If it be a rectangular hyperbola we must have

$$\lambda+\mu-(\nu+\rho)\cos\omega=0,$$

i.e.,
$$\rho=\frac{\lambda}{\cos\omega}+\frac{\mu}{\cos\omega}-\nu,$$

where ω is the angle between the axes of reference. The equilateral hyerbola through (x,y) and (x_1,y_1) is then of the form

$$\frac{\lambda}{\cos\omega}[(X-x_1)\{\cos\omega(X-x)+Y-y\}]$$

$$+\frac{\mu}{\cos\omega}[(Y-y)\{\cos\omega(Y-y_1)+X-x_1\}]$$

$$+\nu[(X-x)(Y-y_1)+(X-x_1)(Y-y)]=0.$$

Hence the equilateral hyperbola through (x,y), (x_1,y_1), (x_2,y_2), (x_3,y_3) is

$$\begin{vmatrix} \{\cos\omega(X-x)+Y-y\}(X-x_1), & \{\cos\omega(Y-y_1)+X-x_1\}(Y-y), & (X-x)(Y-y_1)-(X-x_1)(Y-y) \\ \{\cos\omega(x_2-x)+y_2-y\}(x_2-x_1), & \{\cos\omega(y_2-y_1)+x_2-x_1\}(y_2-y), & (x_2-x)(y_2-y_1)-(x_2-x_1)(y_2-y) \\ \{\cos\omega(x_3-x)+y_3-y\}(x_3-x_1), & \{\cos\omega(y_3-y_1)+x_3-x_1\}(y_3-y), & (x_3-x)(y_3-y_1)-(x_3-x_1)(y_3-y) \end{vmatrix}=0.$$

If (x,y), (x_1,y_1), (x_2,y_2) and (x_3,y_3) are consecutive points, we have

$$x_1=x+dx, \qquad y_1=y+dy,$$

$$x_2=x+2dx+d^2x, \qquad y_2=y+2dy+d^2y,$$

$$x_3=x+3dx+3d^2x+d^3x, \qquad y_3=y+3dy+3d^2y+d^3y.$$

Substituting these values of (x_1,y_1), (x_2,y_2) and (x_3,y_3) and simplifying, we get, neglecting higher orders of differentials

$$\begin{vmatrix} (X-x)^2-(Y-y)^2, & dx(Y-y)-dy(X-x), & \cos\omega(Y-y)^2+(X-x)(Y-y) \\ 2(dx^2-dy^2), & dxd^2y-dyd^2x, & 2\cos\omega(dy)^2+2dydx \\ 6(dxd^2x-dyd^2y), & dxd^3y-dyd^3x, & 6\cos\omega dyd^2y+3dyd^2x+3dxd^2y \end{vmatrix}=0,$$

as the equation of the osculating equilateral hyperbola at the point (x,y). If x be the independent variable $d^2x=0$, $d^3x=0$,... and the above reduces to

$$\begin{vmatrix} (X-x)^2-(Y-y)^2, & Y-y-p(X-x), & \cos\omega(Y-y)^2+(X-x)(Y-y) \\ 2(1-p^2), & q, & 2p^2\cos\omega+2p \\ -6pq, & r, & 6pq\cos\omega+3q \end{vmatrix}=0,$$

or

$$\{(X-x)^2-(Y-y)^2\}(2pr-3q^2)-2(X-x)(Y-y)\{(1-p^2)r+3pq^2\}$$

$$+6\{(Y-y)-p(X-x)\}q(1+p^2)+2\cos\omega[(X-x)^2p(pr-3q^2)$$

$$-r(Y-y)^2+6pq\{Y-y-p(X-x)\}]=0.$$

Putting the values of p, q, r, s from § 1 and simplifying we get for the equation of the osculating equilateral hyperbola

$$(X-x)^2\begin{vmatrix} J_1 & J_2 & 3J & 0 \\ f_{11} & f_{12} & f_1 & f_1 \\ f_{21} & f_{22} & f_2 & 2f_1\cos\omega-f_2 \\ f_1 & f_2 & 0 & 0 \end{vmatrix}$$

$$+(Y-y)^2\begin{vmatrix} J_1 & J_2 & 3J & 0 \\ f_{11} & f_{12} & f_1 & 2f_2\cos\omega-f_1 \\ f_{21} & f_{22} & f_2 & f_2 \\ f_1 & f_2 & 0 & 0 \end{vmatrix}$$

$$+2(X-x)(Y-y)\begin{vmatrix} J_1 & J_2 & 3J & 0 \\ f_{11} & f_{12} & f_1 & f_2 \\ f_{21} & f_{22} & f_2 & f_1 \\ f_1 & f_2 & 0 & 0 \end{vmatrix}$$

$$-6J\begin{vmatrix} Y-y & x-X \\ f_1 & f_2 \end{vmatrix}\begin{vmatrix} 1 & \cos\omega & f_1 \\ \cos\omega & 1 & f_2 \\ f_1 & f_2 & 0 \end{vmatrix}=0,$$

or with the notations of § 1

$$P\xi^2+Q\eta^2+2R\xi\eta-6JS(f_1\xi+f_2\eta)=0 \qquad \dots \quad (1)$$

where $\qquad \xi=X-x$ and $\eta=Y-y$.

For centre we have

$$P\xi+R\eta-3JSf_1=0,$$

$$R\xi+Q\eta-3JSf_2=0;$$

whence

$$\frac{\xi}{3JS(f_1Q-f_2R)}=\frac{\eta}{3JS(f_2P-f_1R)}=\frac{1}{PQ-R^2}.$$

Now $PQ-R^2=-ST$;

$$\therefore\ \xi=-\frac{3J(f_1Q-f_2R)}{T}=\frac{3JMS}{T},$$

$$\eta=-\frac{3J(f_2P-f_1R)}{T}=-\frac{3JNS}{T};$$

i.e.,

$$\left.\begin{aligned} X&=x+\frac{3JMS}{T}, \\ Y&=y-\frac{3JNS}{T}. \end{aligned}\right\} \qquad \ldots \quad (2)$$

3

The general equation of the osculating conic is [1]

$$(3qs-5r^2)\{Y-y-p(X-x)\}^2+\{(Y-y)r-(X-x)(pr-3q^2)\}^2$$
$$=18q^3\{Y-y-p(X-x)\}. \qquad \ldots \quad (3)$$

Now

$$3qs-5r^2=\frac{1}{f_2^{10}}\left\{3J\begin{vmatrix} K_1 & K_2 & 5K \\ f_{21} & f_{22} & f_2 \\ f_1 & f_2 & 0 \end{vmatrix}-5\begin{vmatrix} J_1 & J_2 & 3J \\ f_{21} & f_{22} & f_2 \\ f_1 & f_2 & 0 \end{vmatrix}^2\right\}.$$

$$=\frac{1}{f_2^{10}}\left[3J\{-f_2^2K_1+f_1f_2K_2+5K\begin{vmatrix} f_{21} & f_{22} \\ f_1 & f_2 \end{vmatrix}-5K^2\right]$$

$$=\frac{1}{f_2^{10}}\left[-3f_2^2J\{3J(f_{12}^2+f_2f_{112}-f_{11}f_{22}-f_1f_{122})\right.$$

[1] *Vide*—A General Theory of Osculating Conics, *loc. cit.*

$$+3J_1(f_2f_{12}-f_1f_{22})-2f_1f_{12}J_1-f_2^2J_{11}+f_{11}f_2J_2+f_1f_{12}J_2$$

$$+f_1f_2J_{12}+3f_1f_2J\{(f_{12}f_{12}+f_2f_{122}-f_{11}f_{22}-f_1f_{222})3J+3J_2(f_2f_{12}$$

$$-f_1f_{22})-2f_2f_{22}J_1-f_2^2J_{12}+f_{12}f_2J_2+f_1f_{22}J_2+f_1f_2J_{22}\}$$

$$+15J(f_2f_{12}-f_1f_{22})\{3J(f_2f_{12}-f_1f_{22})-f_2^2J_1+f_1f_2J_2\}$$

$$-5\{9J^2(f_2f_{12}-f_1f_{22})^2+f_2^4J_1^2+f_1^2f_2^2J_2^2-2f_1f_2^3J_1J_2$$

$$-6f_2^2JJ_1(f_2f_{12}-f_1f_{22})+6f_1f_2JJ_2(f_2f_{12}-f_1f_{22})\}\Big]$$

$$=\frac{1}{f_2^{10}}\left[-5f_2^2\begin{vmatrix}J_1 & J_2\\ f_1 & f_2\end{vmatrix}^2-3f_2^2J\begin{vmatrix}J_{11} & J_{12} & f_1\\ J_{21} & J_{22} & f_2\\ f_1 & f_2 & 0\end{vmatrix}\right.$$

$$\left.+12f_2^2J\begin{vmatrix}f_{11} & f_{12} & J_1\\ f_{21} & f_{22} & J_2\\ f_1 & f_2 & 0\end{vmatrix}+27f_2^2J^2\begin{vmatrix}f_{11} & f_{12}\\ f_{12} & f_{22}\end{vmatrix}\right]$$

Putting

$$L=\frac{5}{3}\begin{vmatrix}J_1 & J_2\\ f_1 & f_2\end{vmatrix}^2+J\begin{vmatrix}J_{11} & J_{12} & f_1\\ J_{21} & J_{22} & f_2\\ f_1 & f_2 & 0\end{vmatrix}-4J\begin{vmatrix}f_{11} & f_{12} & J_1\\ f_{21} & f_{22} & J_2\\ f_1 & f_2 & 0\end{vmatrix}-9J^2\begin{vmatrix}f_{11} & f_{12}\\ f_{21} & f_{22}\end{vmatrix}$$

we have

$$3qs-5r^2=-\frac{3L}{f_2^{8}}.$$

Also

$$pr-3q^2=\frac{f_1}{f_2^5}\begin{vmatrix}J_1 & J_2 & 3J\\ f_{21} & f_{22} & f_2\\ f_1 & f_2 & 0\end{vmatrix}-3\frac{J^2}{f_2^5},$$

$$=\frac{1}{f_2^5}\begin{vmatrix}J_1 & J_2 & 3J\\ f_{11} & f_{12} & f_1\\ f_1 & f_2 & 0\end{vmatrix}\text{ on simplification.}$$

Substituting these in (3), the equation transforms into

$$\begin{vmatrix} J_1 & J_2 & 3J & 0 \\ f_{11} & f_{12} & f_1 & Y-y \\ f_{21} & f_{22} & f_2 & x-X \\ f_1 & f_2 & 0 & 0 \end{vmatrix} = 3L \begin{vmatrix} Y- & x-X \\ f_1 & f_2 \end{vmatrix}^2 + 18J^3 \begin{vmatrix} Y-y & x-X \\ f_1 & f_2 \end{vmatrix}$$

which is therefore the equation to the osculating conic at the point (x,y).

The co-ordinates of the centre are

$$X = x - \frac{3qr}{3qs-5r^2},$$

$$Y = y - \frac{3q(pr-3q^2)}{3qs-5r^2},$$

which readily transform into

$$\left.\begin{aligned} X = x - \frac{J}{L} \begin{vmatrix} J_1 & J_2 & 3J \\ f_{21} & f_{22} & f_2 \\ f_1 & f_2 & 0 \end{vmatrix} = x - \frac{JM}{L}, \\ Y = y + \frac{J}{L} \begin{vmatrix} J_1 & J_2 & 3J \\ f_{11} & f_{12} & f_1 \\ f_1 & f_2 & 0 \end{vmatrix} = y + \frac{JN}{L}. \end{aligned}\right\} \quad \ldots \quad (4)$$

4

The axes of the conic are determined as follows.

Putting $\xi = X - x$ and $\eta = Y - y$, the general equation of the osculating conic may be written as

$$(3qs-5r^2)(\eta - p\xi)^2 + \{\eta r - (pr-3q^2)\xi\}^2 - 18q^3(\eta - p\xi) = 0,$$

or $$\xi^2\{p^2\lambda + (pr-3q^2)\} + \eta^2(\lambda + r^2) - 2\xi\eta\{p\lambda + r(pr-3q^2)\}$$
$$-18q^3\eta + 18pq^3\xi = 0,$$

where $$\lambda = 3qs - 5r^2.$$

The axes of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

are given by

$$r^4(ab-h^2)^3-\Delta(ab-h^2)(a+b-2h\cos\omega)r^2+\Delta^2=0$$

where

$$\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},$$

r being a semi-axis of the conic.

Now $\dfrac{\Delta}{ab-h^2}=g\bar{x}+f\bar{y}+c,\quad (\bar{x},\bar{y})$ being the centre.

Hence in our case

$$\frac{\Delta}{ab-h^2}=-9pq^3\,\frac{3qr}{\lambda}+9q^3\,\frac{3q(pr-3q^2)}{\lambda}=-\frac{81q^6}{\lambda};$$

and $\quad ab-h^2=\{p^2\lambda+(pr-3q^2)^2\}(\lambda+r^2)-\{p\lambda+r(pr-3q^2)\}^2=9\lambda q^4.$

Hence if r_1 and r_2 be the semi-axes of the conic

$$r_1^2+r_2^2=\frac{\Delta(a+b-2h\cos\omega)}{(ab-h^2)^2}$$

$$=-\frac{9q^2}{\lambda^2}\,\{\lambda(1+p^2+2p\cos\omega)+(pr-3q^2)^2+2r(pr-3q^2)\cos\omega\}$$

$$=-\frac{J^2}{L^3}\left\{3L\begin{vmatrix} 1 & \cos\omega & f_1 \\ \cos\omega & 1 & f_2 \\ f_1 & f_2 & 0 \end{vmatrix}-\begin{vmatrix} 1 & \cos\omega & M \\ \cos\omega & 1 & N \\ M & N & 0 \end{vmatrix}\right\}$$

$$=\frac{J^2}{L^3}\,(T-3LS)\,;$$

$$r_1^2r_2^2=\frac{\Delta^2}{(ab-h^2)^3}=\left(\frac{\Delta}{ab-h^2}\right)^2\frac{1}{ab-h^2},$$

$$=-27\,\frac{J^8}{L^3}\,.$$

5

The equation to the osculating parabola is[1]

$$\{(X-x)(pr-3q^2)-(Y-y)r\}^2=18q^3\{Y-y-p(X-x)\}$$

which transforms into

$$\begin{vmatrix} J_1 & J_2 & 3J & 0 \\ f_{11} & f_{12} & f_1 & Y-y \\ f_{21} & f_{22} & f_2 & x-X \\ f_1 & f_2 & 0 & 0 \end{vmatrix} = 18J^3 \begin{vmatrix} Y-y & x-X \\ f_1 & f_2 \end{vmatrix}.$$

In rectangular co-ordinates, the semi-latus rectum l is given by

$$l=\frac{27q^5}{\{(pr-3q^2)^2+r^2\}^{\frac{3}{2}}}=\frac{27J^5}{\left[\begin{vmatrix} J_1 & J_2 & 3J \\ f_{11} & f_{12} & f_1 \\ f_1 & f_2 & 0 \end{vmatrix}^2+\begin{vmatrix} J_1 & J_2 & 3J \\ f_{21} & f_{22} & f_2 \\ f_1 & f_2 & 0 \end{vmatrix}^2\right]^{\frac{3}{2}}}$$

$$=\frac{27J^5}{[M^2+N^2]^{\frac{3}{2}}}.$$

The directrix of the osculating parabola is

$$r(X-x)+(pr-3q^2)(Y-y)-\tfrac{3}{2}q(1+p^2)=0,$$

or

$$\begin{vmatrix} J_1 & J_2 & 3J & 0 \\ f_{11} & f_{12} & f_1 & X-x \\ f_{21} & f_{22} & f_2 & Y-y \\ f_1 & f_2 & 0 & 0 \end{vmatrix}+\tfrac{3}{2}J(f_1^2+f_2^2)=0.$$

6

The circle of curvature in rectangular co-ordinates is

$$(X-x)^2+(Y-y)^2=\frac{2(1+p^2)}{q}\{Y-y-p(X-x)\},$$

[1] *Vide*—A General Theory of Osculating Conics.

which transformed is

$$(X-x)^2+(Y-y)^2=\frac{2(f_1^2+f_2^2)}{J}\begin{vmatrix} Y-y & x-X \\ f_1 & f_2 \end{vmatrix}.$$

The co-ordinates of the centre of curvature are

$$X=x-\frac{(1+p^2)p}{q}=x+\frac{f_1(f_1^2+f_2^2)}{J},$$

$$Y=y+\frac{1+p^2}{q}=y+\frac{f_2(f_1^2+f_2^2)}{J};$$

and the radius of curvature ρ is given by

$$\rho=\frac{(1+p^2)^{\frac{3}{2}}}{q}=\frac{(f_1^2+f_2^2)}{J}.$$

7

The osculating conic will be an ellipse, hyperbola or parabola according as $3qs-5r^2$ is positive, negative or zero. But we have seen that $3qs-5r^2=-\frac{3L}{f_2^8}$. Hence the osculating conic will be an ellipse, hyperbola or parabola according as L is negative, positive or zero. Hence the condition [1] for a parabolic point is $L=0$, *i.e.*,

$$\frac{5}{3}\begin{vmatrix} J_1 & J_2 \\ f_1 & f_2 \end{vmatrix}^2+J\begin{vmatrix} J_{11} & J_{12} & f_1 \\ J_{21} & J_{22} & f_2 \\ f_1 & f_2 & 0 \end{vmatrix}-4J\begin{vmatrix} f_{11} & f_{12} & J_1 \\ f_{21} & f_{22} & J_2 \\ f_1 & f_2 & 0 \end{vmatrix}-9J^2\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}=0.$$

The condition that the osculating conic may be an equilateral hyperbola is

$$\lambda(1+p^2)+r^2+(pr-3q^2)^2=0,$$

[1] This condition and the condition for a sextactic point given later were communicated to me, without proof by Prof. Harold Hilton to whom I wish to express my gratitude.

which transforms into

$$3L(f_1^2+f_2^2)=\begin{vmatrix} J_1 & J_2 & 3J \\ f_{11} & f_{12} & f_1 \\ f_1 & f_2 & 0 \end{vmatrix}^2+\begin{vmatrix} J_1 & J_2 & 3J \\ f_{21} & f_{22} & f_2 \\ f_1 & f_2 & 0 \end{vmatrix}^2,$$

or $$3L(f_1^2+f_2^2)=M^2+N^2.$$

8

The condition that the osculating conic may have a six-pointic contact at (x,y) is [1]

$$40r^3-45qrs+9q^2t=0,$$

or $$3q\frac{d}{dx}(3qs-5r^2)-8(3qs-5r^2)\frac{dq}{dx}=0,$$

or $$3J\frac{d}{dx}(L)-8L\frac{d}{dx}(J)=0,$$

or $$3J\left(L_1-\frac{f_1}{f_2}L_2\right)-8L\left(J_1-\frac{f_1}{f_2}J_2\right)=0,$$

where $$L_1=\frac{\partial L}{\partial x} \text{ and } L_2=\frac{\partial L}{\partial y};$$

or $$3J\begin{vmatrix} L_1 & L_2 \\ f_1 & f_2 \end{vmatrix}-8L\begin{vmatrix} J_1 & J_2 \\ f_1 & f_2 \end{vmatrix}=0.$$

My best thanks are due to Prof. S. Mukhopadhyaya for suggestions and encouragement and for the help received from his published works.

[1] *Vide*—A General Theory of Osculating Conics.

3

Certain products involving the divisors of numbers

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(Communicated by Dr. B. Datta)

The products of this note, originating in a generalized Lambert series, are sufficiently curious to deserve a passing notice. By expanding these products for special choices of the arbitrary numerical function $f(n)$ occurring in the exponents of the several factors, it is possible to obtain numerous theorems on partitions of integers. The most of these appear to be too complicated to be of much interest, although some of them have already been obtained otherwise by various writers and have appeared in the literature. A few, however, might prove worth developing.

§1. Let m, n denote integers >0, of which m is odd and n arbitrary. If either of m or n occurs under Σ or Π, the sum or product is with respect to all volues of the m or n as defined. The function $f(y)$ is numerical; that is, for each interger value >0 of $y, f(y)$ takes a single finite value. We shall write

$$f'(n) \equiv f(1)+f(d_1)+f(d_2)+\ldots+f(n),$$

where $1, d_1, d_2, \ldots, n$ are all the divisors of n, and it is assumed that for some $|x|>0$ the series

$$\Sigma f'(n)x^n, =\Sigma \frac{f(n)x^n}{1-x^n} \quad \ldots \quad (A)$$

is absolutely convergent. When for a specific f the convergence condition is $|x|<k$, and the variable x is replaced by $g(y)$, it is further assumed that the condition is transformed, in accordance with the substitution, into $|y|<k'$.

§2. Dividing (A) throughout by x and integrating between the limits 0, x, we get, on taking exponentials of both sides,

$$\Pi(1-x^n)^{f(n)/n}=e^{-\Sigma f'(n)x^n/n} \qquad \dots \quad (1)$$

§3. Let $\mu(n)$ as usual denote the function of Möbius; $\mu(n)=0$ if n is divisible by any square >1, and otherwise $\mu(n)=+1$ or -1 according as n is the product of an even or an odd number of distinct primes. By convention $\mu(1)=1$. The relation between f, f' can be reversed, giving

$$f(n)=\Sigma\mu(d)f'(\delta),$$

where the Σ refers to all pairs (d, δ) of divisors such that $d\delta=n$.

If now we define $f''(n)$ by

$$f''(n)=\Sigma\mu(d)f(\delta),$$

it follows from (1) by a change in notation that

$$\Pi(1-x^n)^{f''(n)/n}=e^{-\Sigma f(n)x^n/n} \qquad \dots \quad (2)$$

The formulas (1), (2) are merely different expressions of one fact; when the function occurring in the exponents under Π is given, we use (1), otherwise (2).

§4. When only plus signs occur in the products we need a theorem stated by Liouville,

$$f_1(m)=-f'(m), \qquad f_1(2n)=2f'(n)-f'(2n),$$

where the definition of $f_1(n)$ is

$$f_1(n)=\Sigma(-1)^{\delta} f(d),$$

the Σ extending as before to all pairs (d, δ) of conjugate divisors of n.

To prove this, assume that for some s the Dirichlet series

$$\Sigma\frac{(-1)^n}{n^s}, \qquad \Sigma\frac{f(n)}{n^s},$$

are both absolutely convergent. On multiplying these together and using the identity

$$1+\frac{1}{3^s}+\frac{1}{5^s}+\ldots=\left(1-\frac{1}{2^s}\right)\left(1+\frac{1}{2^s}+\frac{1}{3^s}+\ldots\right),$$

we note that the coefficient of $\frac{1}{n^s}$, which evidently is $f_1(n)$, is as stated by the theorem.

Proceeding as in §2 we get

$$\Pi(1+x^n)^{f(n)/n}=e^{-\Sigma f_1(n)x^n/n} \qquad \ldots \quad (3)$$

§5. When only odd integers m occur in the product we require $f_2(n)$, defined by

$$f_2(n)=\Sigma f(t),$$

the Σ extending to all *odd* divisors t of m. Then

$$\Sigma\frac{f(m)x^m}{1-x^m}=\Sigma f_2(n)x^n,$$

and as before we infer

$$\Pi(1-x^m)^{f(m)/m}=e^{-\Sigma f_2(n)x^n/x^n} \qquad \ldots \quad (4)$$

§6. Obviously (3) can be obtained from (1) by changing x into x^2 and dividing the resulting identity member by member by (1).

The result of changing x into $-x$ in (4) is

$$\Pi(1+x^m)^{f(m)/m}=e^{-\Sigma f_2(2n)x^{2n}/2n}\cdot e^{\Sigma f'(m)x^m/m}. \qquad \ldots \quad (5)$$

Hence by division we get from (4), (5),

$$\Pi\left(\frac{1-x^m}{1+x^m}\right)^{f(m)/m}=e^{-\Sigma f'(m)x^m/m}. \qquad \ldots \quad (6)$$

§7. As the simplest illustrations of (1)-(6) we may take $f(n)=n$ for all integers n. Then $f'(n)=\sigma(n)$, the sum of the divisors of n, and (1) becomes

$$\Pi(1-x^n)=e^{-\Sigma\sigma(n)x^n/n}, \qquad \ldots \quad (1\cdot1)$$

which is merely a restatement, in its simplest reading, of a well-known relation between partitions and divisors. For this choice of f the rest can be interpreted in similar ways

For the choice $f(n)=1$ or 0 according as n is or is not the integer $r>0$, (2) gives

$$\Pi(1-x^{rn})^{\mu(n)/n}=e^{-x^r}, \qquad \dots \quad (2\cdot 1)$$

and hence, for $r=1$,

$$\Pi(1-x^n)^{\mu(n)/n}=e^{-x}. \qquad \dots \quad (2\cdot 12)$$

In all such examples convergence conditions upon x can be readily determined from the corresponding series as indicated in §2.

If $f(n)\equiv\lambda(n)=+1$ or -1 according as the *total* number of prime divisors of n is even or odd, we have $f'(n)=1$ or 0 according as n is or is not a square. Hence (1) gives

$$\Pi(1-x^n)^{\lambda(n)/n}=e^{-\Sigma x^{n^2}/n^2}. \qquad \dots \quad (1\cdot 2)$$

For the choice $f(n)=1$, we get identities involving the number $v(n)$ of divisors of n; $f'(n)$ in this case reduces to 1 for $n=1$, to zero for $n>1$, providing a check.

Identities concerning partitions into primes exclusively (or into any other class of numbers, by the appropriate choice of f), can be obtained by expanding the exponential function and developing the product, as in

$$\Pi'\frac{1}{1-x^p}=e^{\Sigma\hat{\omega}(n)x^n/n} \qquad \dots \quad (1\cdot 3)$$

where $\hat{\omega}(n)$ is the sum of the distinct prime divisors $\gtrless 1$ of n, and Π' refers to all primes $p\gtrless 1$.

As a last example take $f(n)=\phi_r(n)$, where $\phi_r(n)$ is Jordan's totient of order r; $\phi_r(n)$ the number of sets of r equal or distinct integers equal to or less than n and coprime with n. Then $\phi_1(n)=\phi(n)$, Euler's function. It is well-known that $\Sigma\phi_r(d)=n^r$. Let θ denote the operation $x\dfrac{d}{dx}$, *viz.*, the x-derivative of the operand is to be multiplied by x, and θ^r means θ operating upon the result of θ^{r-1}. Then if $\theta^0=1$,

$$\Sigma n^r x^n=\theta^r\Sigma x^n=\theta^r\left(\frac{x}{1-x}\right)\equiv F_r(x),$$

and from (1) we get

$$\Pi(1-x^n)^{\phi_r(n)/n} = e^{-F_{r-1}(x)} \qquad (r \geqq 1). \quad \dots \quad (1\cdot 4)$$

For example taking $r=1$, $x=\frac{1}{2}$,

$$\Pi\left(1-\frac{1}{2^n}\right)^{\phi(n)/n} = \frac{1}{e}\ ; \qquad \dots \quad (1\cdot 41)$$

for $r=2$, $x=\frac{1}{2}$

$$\Pi\left(1-\frac{1}{2^n}\right)^{\phi_2(n)/n} = \frac{1}{e^2}; \qquad \dots \quad (1\cdot 42)$$

for $r=3$, $x=\frac{1}{2}$,

$$\Pi\left(1-\frac{1}{2^n}\right)^{\phi_3(n)/n} = \frac{1}{e^6};$$

and so on.

University of Washington,
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ON LIQUID MOTION INSIDE CERTAIN ROTATING CIRCULAR ARCS

BY

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In the present paper, I have discussed the problem of liquid motion in rotating vessels when the boundary consists of (1) four orthogonal circles, (2) three circles, one cutting the other two orthogonally (3) two orthogonal circles. I have also deduced some particular cases of boundaries consisting of arcs of circles and straight lines.

1. Let $$x+iy=c\tan\tfrac{1}{2}(\xi+i\eta)$$

so that ξ=const and η=const represent two systems of co-axial circles cutting orthogonally.

If ψ be the stream function, and ω the angular velocity of the cylinder, then we have

$$\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial\eta^2}=0$$

at all points in the liquid and

$$\psi=\tfrac{1}{2}\omega(x^2+y^2)+\text{const}$$

at all points on the boundary.

Evidently,

$$y=c+2c\sum_{n=1}^{\infty}(-1)^n e^{-n\eta}\cos n\xi,$$

$$x=2c\sum_{n=1}^{\infty}(-1)^{n+1}e^{-n\eta}\sin n\xi.$$

On the boundary ξ=const,

$$\psi=-c\omega x\cot\xi+\text{const}$$

$$=2c^2\omega\cot\xi\sum_{1}^{\infty}(-1)^n e^{-n\eta}\sin n\xi+\text{const.}$$

On the boundary $\eta=\text{const},$

$$\psi=c\omega y\coth\eta+\text{const.}$$

$$=2c^2\omega\coth\eta\sum_1^\infty(-1)^n e^{-n\eta}\cos n\xi+\text{const.}$$

Rectangle bounded by four circular arcs

2. Let the section of the cylinder be bounded by the circles

$$\xi=\alpha_1,\ \xi=\alpha_2,\ \eta=\beta_1 \text{ and } \eta=\beta_2.$$

Assume

$$\psi=2c^2\omega\sum_1^\infty(-1)^n\frac{e^{-n\beta_2}\coth\beta_2\sinh n(\eta-\beta_1)+e^{-n\beta_1}\coth\beta_1\sinh n(\beta_2-\eta)}{\sinh n(\beta_2-\beta_1)}\times\cos n\xi$$

$$+2c^2\omega\sum_1^\infty(-1)^n\frac{\sin n\alpha_2\cot\alpha_2\sin n(\xi-\alpha_1)+\sin n\alpha_1\cot\alpha_1\sin n(\alpha_2-\xi)}{\sin n(\alpha_2-\alpha_1)}e^{-n\eta}$$

$$+\sum_{m=0}^\infty\left\{P_m\sinh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-\alpha_1)\right.$$

$$\left.+Q_m\sinh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-\alpha_2)\right\}\cos\frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta)$$

$$+\sum_{m=0}^\infty\left\{R_m\sinh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\beta_1)\right.$$

$$\left.+S_m\sinh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\beta_2)\right\}\cos\frac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma)\quad\ldots\quad(1)$$

where

$$2\epsilon_1=\alpha_2-\alpha_1\qquad 2\gamma=\alpha_2+\alpha_1$$

$$2\epsilon_2=\beta_2-\beta_1,\qquad 2\delta=\beta_2+\beta_1\quad\ldots\quad(2)$$

When $\xi = \alpha_1$ and $\beta_1 \leq \eta \leq \beta_2$, we have

$$2c^2\omega \cot \alpha_1 \sum_1^\infty (-1)^n e^{-n\eta} \sin n\alpha_1$$

$$= 2c^2\omega \cot \alpha_1 \sum_1^\infty (-1)^n e^{-n\eta} \sin n\alpha_1$$

$$+ 2c^2\omega \sum_1^\infty (-1)^n \frac{(e^{-n\beta_2} \coth \beta_2 - e^{-n\beta_1} \coth \beta_1) \cosh n\epsilon_2 \sinh n(\eta-\delta) + (e^{-n\beta_2} \coth \beta_2 + e^{-n\beta_1} \coth \beta_1) \sinh n\epsilon_2 \cosh n(\eta-\delta)}{\sinh 2n\epsilon_2} \cos n\alpha_1$$

$$- \sum_{m=0}^\infty Q_m \sinh \frac{(2m+1)\pi\epsilon_1}{\epsilon_2} \cos \frac{(2m+1)\pi}{2\epsilon_2} (\eta - \delta).$$

Multiplying both sides by $\cos \frac{(2m+1)\pi}{2\epsilon_2} (\eta-\delta)\, d\eta$ and integrating between β_1 and β_2 and simplifying we have

$$Q_m = \frac{4c^2\omega}{\sinh \frac{(2m+1)\pi\epsilon_1}{\epsilon_2}} \sum_{n=1}^\infty (-1)^{m+n} \cdot \frac{(2m+1)\pi}{(2m+1)^2\pi^2 + 4n^2\epsilon_2^2} \cdot (e^{-n\beta_2} \coth \beta_2 + e^{-n\beta_1} \coth \beta_1) \cos n\alpha_1. \quad \ldots \quad (3)$$

Putting $\xi = \alpha_2$ and proceeding in the same way as above we have

$$P_m = \frac{4c^2\omega}{\sinh \frac{(2m+1)\pi\epsilon_1}{\epsilon_2}} \sum_{n=1}^\infty (-1)^{m+n+1} \cdot \frac{(2m+1)\pi}{(2m+1)^2\pi^2 + 4n^2\epsilon_2^2} (e^{-n\beta_2} \coth \beta_2 + e^{-n\beta_1} \coth \beta_1) \cos n\alpha_2. \quad \ldots \quad (4)$$

From the boundary condition on $\eta=\beta_1$, we have, by multiplying both sides by $\cos\dfrac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma)d\xi$ and integrating between a_1 and a_2

$$S_m=\frac{4c^2\omega}{\sinh\dfrac{(2m+1)\pi\epsilon_2}{\epsilon_1}}\sum_{n=1}^{\infty}(-1)^{m+n}\frac{(2m+1)\pi}{(2m+1)^2\pi^2-4n^2\epsilon_1^{\,2}}(\sin na_2\cot a_2+\sin na_1\cot a_1)e^{-n\beta_1}. \quad \dots (5)$$

Similarly putting $\eta=\beta_2$ and proceeding as above we have

$$R_m=\frac{4c^2\omega}{\sinh\dfrac{(2m+1)\pi\epsilon_2}{\epsilon_1}}\sum_{n=1}^{\infty}(-1)^{m+n+1}\frac{(2m+1)\pi}{(2m+1)^2\pi^2-4n^2\epsilon^2_{\,1}}(\sin na_2\cot a_2+\sin na_1\cot a_1)e^{-n\beta_2}. \quad \dots (6)$$

Thus ψ is completely determined.

$$\therefore\ \phi=2c^2\omega\sum_1^{\infty}(-1)^n\frac{e^{-n\beta_2}\coth\beta_2\cosh n(\eta-\beta_1)-e^{-n\beta_1}\coth\beta_1\cosh n(\beta_2-\eta)}{\sinh n(\beta_2-\beta_1)}\sin n\xi$$

$$+2c^2\omega\sum_1^{\infty}(-1)^n\frac{\sin na_2\cot a_2\cos n(\xi-a_1)-\sin na_1\cot a_1\cos n(a_2-\xi)}{\sin n(a_2-a_1)}e^{-n\eta}$$

$$-\sum_{m=0}^{\infty}\left\{P_m\cosh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-a_1)+Q_m\cosh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-a_2)\right\}\sin\frac{(2m+1)\pi}{2\epsilon_2}(n-\delta)$$

$$+\sum_{m=0}^{\infty}\left\{R_m\cosh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\beta_1)+S_m\cosh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\beta_2)\right\}\sin\frac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma). \quad \dots (7)$$

If we put
$$\psi_1=\sum_{m=0}^{\infty}\left\{P_m\sinh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-\alpha_1)+Q_m\sinh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-\alpha_2)\right\}\cos\frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta)$$
$$+\sum_{m=0}^{\infty}\left\{R_m\sinh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\beta_1)+S_m\sinh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\beta_2)\right\}\cos\frac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma)\,; \qquad \ldots \quad (8)$$
then the part ψ_1' of ψ can be proved to be
$$=2c^2\omega\sum_{m=0}^{\infty}(-1)^m\left[\frac{\sinh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-\gamma)}{\sinh\frac{(2m+1)\pi\epsilon_1}{2\epsilon_2}}\sum_{n=1}^{\infty}(-1)^n\frac{(2m+1)\pi}{(2m+1)^2\pi^2+4n^2\epsilon^2{}_2}A_n\right.$$
$$\left.+\frac{\cosh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-\gamma)}{\cosh\frac{(2m+1)\pi\epsilon_1}{2\epsilon_2}}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2m+1)\pi}{(2m+1)^2\pi^2+4n^2\epsilon^2{}_2}B_n\right]\cos\frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta)$$
$$+2c^2\omega\sum_{m=0}^{\infty}(-1)^m\left[\frac{\sin\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\delta)}{\sinh\frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\sum_{n=1}^{\infty}(-1)^n\frac{(2m+1)\pi}{(2m+1)^2\pi^2-4n^2\epsilon^2{}_1}C_n\right.$$
$$\left.+\frac{\cosh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\delta)}{\cosh\frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2m+1)\pi}{(2m+1)^2\pi^2-4n^2\epsilon^2{}_1}D_n\right]\cos\frac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma)$$

when

$$\left.\begin{aligned}
A_n &= (e^{-n\beta_2}\coth\beta_2 + e^{-n\beta_1}\coth\beta_1)(\cos n\alpha_1 - \cos n\alpha_2)\\
B_n &= (e^{-n\beta_2}\coth\beta_2 + e^{-n\beta_1}\coth\beta_1)(\cos n\alpha_1 + \cos n\alpha_2)\\
C_n &= (\sin n\alpha_2 \cot\alpha_2 + \sin n\alpha_1 \cot\alpha_1)(e^{-n\beta_1}\ e^{-n\beta_2})\\
D_n &= (\sin n\alpha_2 \cot\alpha_2 + \sin n\alpha_1 \cot\alpha_1)(e^{-n\beta_1} + e^{-n\beta_2})
\end{aligned}\right\} \quad \ldots \quad (9)$$

$$\therefore\ \psi_1 = 2c^2\omega \sum_{n=1}^{\infty}(-1)^n\Bigg[A_n \sum_{m=0}^{\infty}(-1)^m \frac{(2m+1)\pi}{(2m+1)^2\pi^2+4n^2\epsilon^2_2}\cdot\frac{\sinh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-\gamma)}{\sinh\frac{(2m+1)\pi\epsilon_1}{2\epsilon_2}}\cos\frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta)$$

$$+C_n \sum_{m=0}^{\infty}(-1)^m \frac{(2m+1)\pi}{(2m+1)^2\pi^2-4n^2\epsilon^2_1}\cdot\frac{\sinh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\delta)}{\sinh\frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\cos\frac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma)\Bigg]$$

$$-2c^2\omega \sum_{n=1}^{\infty}(-1)^n\Bigg[B_n \sum_{m=0}^{\infty}(-1)^m \frac{(2m+1)\pi}{(2m+1)^2\pi^2+4n^2\epsilon^2_2}\cdot\frac{\cosh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-\gamma)}{\cosh\frac{(2m+1)\pi\epsilon_1}{2\epsilon_2}}\cos\frac{(2m+1)\mu}{2\epsilon_2}(\eta-\delta)$$

$$+D_n \sum_{m=0}^{\infty}(-1)^m \frac{(2m+1)\pi}{(2m+1)^2\pi^2-4n^2\epsilon^2_1}\cdot\frac{\cosh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\delta)}{\cosh\frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\cos\frac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma)\Bigg]$$

$$\text{Let } u=\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\left\{\frac{\cosh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-\gamma)}{\cosh\frac{(2m+1)\pi\epsilon_1}{2\epsilon_2}}\cos\frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta)+\frac{\cosh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\delta)}{\cosh\frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\cos\frac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma)\right\}. \quad (10)$$

When $\xi=\alpha_1$ or α_2 and $\beta_2 \geq \eta \geq \beta_1$, $\quad u=\sum\frac{(-1)^m}{2m+1}\cos\frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta) \quad =\frac{\pi}{4}$.

Similarly when $\eta=\beta_1$ or β_2 and $\alpha_2 \geq \xi \geq \alpha_1$ $\quad u=\frac{\pi}{4}$.

Also $\qquad \frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}=0 \qquad \therefore\ u=\frac{\pi}{4}$ within the boundaries.

Since $\qquad \frac{\partial^2 u}{\partial \xi^2}=-\frac{\partial^2 u}{\partial \eta^2} \qquad \therefore \left(\frac{\partial^2}{\partial \xi^2}+n^2\right)^{-1}u=\left(-\frac{\partial^2}{\partial \eta^2}+n^2\right)^{-1}u$

Performing these operations on both sides of u we have

$$\frac{\pi}{4n^2}=\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\left\{\frac{\cosh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-\gamma)}{\cosh\frac{(2m+1)\pi\epsilon_1}{2\epsilon_2}}\cdot\frac{\cos\frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta)}{\frac{(2m+1)^2\pi^2}{4\epsilon_2^2}+n^2}\right.$$

$$\left.-\frac{\cosh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\delta)}{\cosh\frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\cdot\frac{\cos\frac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma)}{\frac{(2m+1)^2\pi^2}{4\epsilon_1^2}-n^2}\right\}+f_1(\eta)\cos n(\xi-\gamma)$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\left\{\frac{\cosh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-\gamma)}{\cosh\frac{(2m+1)\pi\epsilon_1}{2\epsilon_2}}\cdot\frac{\cos\frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta)}{\frac{(2m+1)^2\pi^2}{4\epsilon_2^2}+n^2}\right.$$

$$\left.-\frac{\cosh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\delta)}{\cosh\frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\cdot\frac{\cos\frac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma)}{\frac{(2m+1)^2\pi^2}{2\epsilon_1^2}-n^2}\right\}+f_2(\xi)\cosh n(\eta-\delta)$$

where $\alpha_2>\xi>\alpha_1$ and $\beta_2>\eta>\beta_1$ $\quad\therefore\ f_2(\xi)=c\cos n(\xi-\gamma)$ and $f_1(\eta)=c\cosh n(\eta-\delta)$.

Putting $\xi=\alpha_2$ and $\eta=\beta_2$ we have $\quad C=\dfrac{\pi}{4n^2\cos n\epsilon_1.\cosh n\epsilon_2}$.

$$\therefore\ \frac{\pi}{4}\left[1-\frac{\cos n(\xi-\gamma)\cosh n(\eta-\delta)}{\cos n\epsilon_1\cosh n\epsilon_2}\right]=\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\left\{\frac{4n^2\epsilon_2^2}{(2m+1)^2\pi^2+4n^2\epsilon_2^2}\cdot\frac{\cosh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-\gamma)}{\cosh\frac{(2m+1)\pi\epsilon_1}{2\epsilon_2}}\cos\frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta)\right.$$

$$\left.-\frac{4n^2\epsilon_1^2}{(2m+1)^2\pi^2-4n^2\epsilon_1^2}\cdot\frac{\cosh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\delta)}{\cosh\frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\cos\frac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma)\right\}.\quad\ldots\quad(11)$$

Subtracting (11) from (10) and dividing by π we have

$$\frac{\cos n(\xi-\gamma)\cosh n(\eta-\delta)}{4\cos n\epsilon_1\cosh n\epsilon_2} = \sum_{m=0}^{\infty}(-1)^m\left\{\frac{(2m+1)\pi}{(2m+1)^2\pi^2+4n^2\epsilon_2{}^2}\cdot\frac{\cosh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-\gamma)}{\cosh\frac{(2m+1)\pi\epsilon_1}{2\epsilon_2}}\cos\frac{(2m+1)\pi}{2\epsilon_2}(\eta-\delta)\right.$$

$$\left.+\frac{(2m+1)\pi}{(2m+1)^2\pi^2-4n^2\epsilon_1{}^2}\cdot\frac{\cosh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\delta)}{\cosh\frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\cos\frac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma)\right\}. \quad \ldots \quad (12)$$

$$\therefore\ \psi_1=2c^2\omega\sum_{n=1}^{\infty}(-1)^n\left[A_n\sum_{m=0}^{\infty}(-1)^m\frac{(2m+1)\pi}{(2m+1)^2\pi^2+4n^2\epsilon^2{}_2}\cdot\frac{\sinh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-\gamma)}{\sinh\frac{(2m+1)\pi\epsilon_1}{2\epsilon_1}}\cos\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\delta)\right.$$

$$\left.+C_n\sum_{m=0}^{\infty}(-1)^m\frac{(2m+1)\pi}{(2m+1)^2\pi^2-4n^2\epsilon_1{}^2}\cdot\frac{\sinh\frac{(2m+1)\pi}{2\epsilon_2}(\xi-\gamma)}{\sinh\frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\cos\frac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma)\right]$$

$$-\frac{1}{2}c^2\omega\sum_{n=1}^{\infty}(-1)^n B_n\frac{\cos n(\xi-\gamma)\cosh n(\eta-\delta)}{\cos n\epsilon_1\cosh n\epsilon_2}$$

$$-2c^2\omega\sum^{\infty}(-1)^n(B_n-D_n)\sum^{\infty}(-1)^m\frac{(2m+1)\pi}{(2m+1)^2\pi^2-4n^2\epsilon_1{}^2}\cdot\frac{\cosh\frac{(2m+1)\pi}{2\epsilon_1}(\eta-\delta)}{\cosh\frac{(2m+1)\pi\epsilon_2}{2\epsilon_1}}\cos\frac{(2m+1)\pi}{2\epsilon_1}(\xi-\gamma). \quad \ldots \quad (13)$$

Another expression for ψ can be obtained by assuming

$$\psi=2c^2\omega\sum_1^\infty(-1)^n\frac{e^{-n\beta_2}\coth\beta_2\sinh n(\eta-\beta_1)+e^{-n\beta_1}\coth\beta_1\sinh n(\beta_2-\eta)}{\sinh n(\beta_2-\beta_1)}\cos n\xi$$

$$+2c^2\omega\sum_1^\infty(-1)^n\frac{\sin na_2\cot a_2\sin n(\xi-a_1)+\sin na_1\cot a_1\sin n(a_2-\xi)}{\sin n(a_2-a_1)}e^{-n\eta}$$

$$+\sum_1^\infty\left\{P_m\sinh\frac{m\pi}{\epsilon_2}(\xi-a_1)+Q_m\sinh\frac{m\pi}{\epsilon_2}(\xi-a_2)\right\}\sin\frac{m\pi}{\epsilon_2}(\eta-\delta)$$

$$+\sum_1^\infty\left\{R_m\sinh\frac{m\pi}{\epsilon_1}(\eta-\beta_1)+S_m\sinh\frac{m\pi}{\epsilon_1}(\eta-\beta_2)\right\}\sin\frac{m\pi}{\epsilon_1}(\xi-\gamma).$$

It will be found that $$P_m=\frac{2c^2\omega}{\sinh\frac{2m\pi\epsilon_1}{\epsilon_2}}\sum_{n=1}^\infty(-1)^{m+n}\frac{m\pi}{m^2\pi^2+n^2\epsilon_2^2}\left(e^{-n\beta_2}\coth\beta_2-e^{-n\beta_1}\coth\beta_1\right)\cos na_2$$

$$Q_m=\frac{2c^2\omega}{\sinh\frac{2m\pi\epsilon_1}{\epsilon_2}}\sum_{n=1}^\infty(-1)^{m+n+1}\frac{m\pi}{m^2\pi^2+n^2\epsilon_2^2}\left(e^{-n\beta_2}\coth\beta_2-e^{-n\beta_1}\coth\beta_1\right)\cos na_1$$

$$R_m=\frac{2c^2\omega}{\sinh\frac{2m\pi\epsilon_2}{\epsilon_1}}\sum_{n=1}^\infty(-1)^{m+n}\frac{m\pi}{m^2\pi^2-n^2\epsilon_1^2}\left(\sin na_2\cot a_2-\sin na_1\cot a_1\right)e^{-n\beta_2}$$

$$S_m=\frac{2c^2\omega}{\sinh\frac{2m\pi\epsilon_2}{\epsilon_1}}\sum_{n=1}^\infty(-1)^{m+n+1}\frac{m\pi}{m^2\pi^2-n^2\epsilon_1^2}\left(\sin na_2\cot a_2-\sin na_1\cot a_1\right)e^{-n\beta_1}$$

Triangle bounded by three circles

3. Let the cylinder be bounded by $\xi=\alpha_1$, $\xi=\alpha_2$ and $\eta=\beta$.

At the point where $\xi=\alpha_1$ and $\xi=\alpha_2$ intersect, we have $\eta=\infty$.

Assume

$$\psi=2c^2\omega\sum_{n=1}^{\infty}(-1)^n\coth\beta e^{-n\eta}\cos n\xi$$

$$+2c^2\omega\sum_{n=1}^{\infty}(-1)^n\frac{\sin n\alpha_2\cot\alpha_2\sin n(\xi-\alpha_1)+\sin n\alpha_1\cot\alpha_1\sin n(\alpha_2-\xi)}{\sin n(\alpha_2-\alpha_1)}e^{-n\eta}$$

$$+c^2\omega\sum_{m=0}^{\infty}P_m\cos\frac{(2m+1)\pi}{2\epsilon}(\xi-\gamma)\,e^{-\frac{(2m+1)\pi}{2\epsilon}(\eta-\beta)}$$

$$+c^2\omega\sum_{n=1}^{\infty}\left\{R_n\int_0^{\infty}e^{-n\lambda}d\lambda\int_0^{\infty}\frac{\sinh a(\xi-\alpha_1)}{\sinh 2a\epsilon}\sin a\lambda\sin a(\eta-\beta)da\right.$$

$$\left.+S_n\int_0^{\infty}e^{-n\lambda}d\lambda\int_0^{\infty}\frac{\sinh a(\xi-\alpha_2)}{\sinh 2a\epsilon}\sin a\lambda\sin a(\eta-\beta)da\right\}, \quad\ldots\quad (1)$$

where $\qquad 2\gamma=\alpha_2+\alpha_1,\ 2\epsilon=\alpha_2-\alpha_1;\qquad\ldots\quad(2)$

and the form of a is such that $a(\eta-\beta)$ is a multiple of π when $\eta=\infty$

when $\qquad \eta=\infty$ we have $\psi=0$

when $\qquad \eta=\beta$ and $\alpha_2\geq\xi\geq\alpha_1$

$$2c^2\omega\coth\beta\sum_1^{\infty}(-1)^ne^{-n\beta}\cos n\xi=2c^2\omega\coth\beta\sum_1^{\infty}(-1)^ne^{-n\beta}\cos n\xi$$

$$+2c^2\omega\sum_1^{\infty}(-1)^n\frac{\sin n\alpha_2\cot\alpha_2\sin n(\xi-\alpha_1)+\sin n\alpha_1\cot\alpha_1\sin n(\alpha_2-\xi)}{\sin n(\alpha_2-\alpha_1)}e^{-n\beta}$$

$$+c^2\omega\sum_{m=0}^{\infty}P_m\cos\frac{(2m+1)\pi}{2\epsilon}(\xi-\gamma).$$

Multiplying both sides by

$$\cos\frac{(2m+1)\pi}{2\epsilon}(\xi-\gamma)d\xi$$

and integrating between the limits a_1 and a_2 we have

$$P_n = \sum_{n=1}^{\infty} (-1)^{m+n+1} \frac{4(2m+1)\pi}{(2m+1)^2\pi^2 - 4n^2\epsilon^2} (\sin na_2 \cot a_2 + \sin na_1 \cot a_2) e^{-n\beta} \tag{3}$$

when $\xi = a_1$ and η lies between β and ∞

$$2c^2\omega \cot a_1 \sum_1^{\infty} (-1)^n e^{-n\eta} \sin na_1$$

$$= 2c^2\omega \cot a_1 \sum_1^{\infty} (-1)^n e^{-n\eta} \sin na_1$$

$$+ 2c^2\omega \sum_1^{\infty} (-1)^n e^{-n\eta} \coth\beta \cos na_1$$

$$- c^2\omega \sum_{n=1}^{\infty} S_n \int_0^{\infty} e^{-n\lambda} d\lambda \int_0^{\infty} \sin a\lambda \sin a(\eta-\beta) da.$$

But for positive values of $\eta - \beta$

$$\int_0^{\infty} e^{-n\lambda} d\lambda \int_0^{\infty} \sin a\lambda \sin a(\eta-\beta)\, da = \frac{\pi}{2} e^{-n(\eta-\beta)}$$

$\therefore$ we have $\quad S_n = (-1)^n \frac{4}{\pi} e^{-n\beta} \coth\beta \cos na_1. \quad \ldots \quad (4)$

Similarly we have, from condition on the boundary $\xi = a_2$

$$R_n = (-1)^{n+1} \frac{4}{\pi} e^{-n\beta} \coth\beta \cos na_2. \quad \ldots \quad (5)$$

To evaluate the integrals

$$I_1 = \int_0^{\infty} e^{-n\lambda} d\lambda \int_0^{\infty} \frac{\sinh a(\xi - a_1)}{\sinh 2a\epsilon} \sin a\lambda \sin a(\eta-\beta) da$$

$$= \frac{1}{2} \int_0^{\infty} e^{-n\lambda} d\lambda \int_0^{\infty} \frac{\sinh a(\xi-a_1)}{\sinh 2a\epsilon} \{\cos a(\lambda-\eta+\beta) - \cos a(\lambda+\eta-\beta)\} da$$

But we have [1]

$$\int_0^\infty \frac{\sinh px}{\sinh qx}\cos rx\,dx=\frac{\pi}{2q}\cdot\frac{\sin\frac{p\pi}{q}}{\cos\frac{p\pi}{q}+\cosh\frac{r\pi}{q}},\quad \text{if } p^2<q^2$$

$$\therefore\ I_1=\frac{1}{2}\int_0^\infty e^{-n\lambda}d\lambda\Big[\frac{\pi}{4\epsilon}\cdot\frac{\sin\frac{\pi}{2\epsilon}(\xi-a_1)}{\cos\frac{\pi}{2\epsilon}(\xi-a_1)+\cosh\frac{\pi}{2\epsilon}(\lambda-\eta+\beta)}$$

$$-\frac{\pi}{2\epsilon}\cdot\frac{\sin\frac{\pi}{2\epsilon}(\xi-a_1)}{\cos\frac{\pi}{2\epsilon}(\xi-a_1)+\cosh\frac{\pi}{2\epsilon}(\lambda+\eta-\beta)}\Big]$$

$$\frac{\sin\frac{\pi}{2\epsilon}(\xi-a_1)}{\cos\frac{\pi}{2\epsilon}(\xi-a_1)+\cosh\frac{\pi}{2\epsilon}(\lambda+\eta-\beta)}=\frac{\sin\theta_1}{\cos\theta_1+\cosh x}$$

where

$$\theta_1=\frac{\pi}{2\epsilon}(\xi-a_1)\text{ and }x=\frac{\pi}{2\epsilon}(\lambda+\eta-\beta).$$

Since λ lies between 0 and ∞ and $\eta-\beta$ is positive therefore x is positive. Now,

$$\frac{\sin\theta_1}{\cos\theta_1+\cosh x}=2e^{-\lambda}\sin\theta_1\,[1+\Sigma A_n e^{-nx}].$$

where

$$A_n=(-1)^n[e^{ni\theta_1}+e^{-ni\theta_1}+e^{(n-2)i\theta_1}+e^{-(n-2)i\theta_1}+\ldots]$$

$$=(-1)^n 2[\cos n\theta_1+\cos(n-2)\theta_1+\ldots]$$

$$A_n\sin\theta_1=(-1)^n 2\sin\theta_1[\cos n\theta_1+\cos(n-2)\theta_1+\ldots]$$

$$=(-1)^n\sin(n+1)\theta_1$$

$$\therefore\ \frac{\sin\theta_1}{\cos\theta_1+\cosh x}=2\sum_{m=1}^{\infty}(-1)^{m+1}e^{-m\lambda}\sin m\theta_1$$

[1] Bierens de Haan, Tables of Def. Int. (7) 265.

$$\therefore \int_0^\infty \frac{\sin\frac{\pi}{2\epsilon}(\xi-a_1)}{\cos\frac{\pi}{2\epsilon}(\xi-a_1)+\cosh\frac{\pi}{2\epsilon}(\lambda+\eta-\beta)}\, e^{-n\lambda}\, d\lambda$$

$$=2\sum_{m=1}^{\infty}(-1)^{m+1}\int_0^\infty e^{-n\lambda}\, d\lambda \cdot e^{-\frac{m\pi}{2\epsilon}(\lambda+\eta-\beta)} \sin\frac{m\pi}{2\epsilon}(\xi-a_1)$$

$$=2\sum_{m=1}^{\infty}(-1)^{m+1}\frac{e^{-\frac{m\pi}{2\epsilon}(\eta-\beta)}}{n+\frac{m\pi}{2\epsilon}}\sin\frac{m\pi}{2\epsilon}(\xi-a_1)$$

$$\int_0^\infty \frac{\sin\theta_1 e^{-n\lambda}\, d\lambda}{\cos\theta_1+\cosh\frac{\pi}{2\epsilon}(\lambda-\eta+\beta)}=\int_{\eta-\beta}^\infty \frac{\sin\theta_1 e^{-n\lambda}\, d\lambda}{\cos\theta_1+\cosh\frac{\pi}{2\epsilon}(\lambda-\eta+\beta)}$$

$$+\int_0^{\eta-\beta}\frac{\sin\theta_1 e^{-n\lambda}\, d\lambda}{\cos\theta_1+\cosh\frac{\pi}{2\epsilon}(\lambda-\eta+\beta)}.$$

In the first integral on the right hand side $\lambda-\eta+\beta$ is positive and in the second $\lambda-\eta+\beta$ is negative.

Put $$x=\frac{\pi}{2\epsilon}(\lambda-\eta+\beta)$$

when x is positive

$$\frac{\sin\theta_1}{\cos\theta_1+\cosh x}=2\sum_{m=1}^{\infty}(-1)^{m+1}e^{-mx}\sin m\theta_1$$

when x is negative

$$\frac{\sin\theta_1}{\cos\theta_1+\cosh x}=2\sum_{m=1}^{\infty}(-1)^{m+1}e^{mx}\sin m\theta_1\ ;$$

$$\int_{\eta-\beta}^{\infty} \frac{\sin\theta_1 e^{-n\lambda}\, d\lambda}{\cos\theta_1 + \cosh\frac{\pi}{2\epsilon}(\lambda-\eta+\beta)}$$

$$=2\sum_{m=1}^{\infty}(-1)^{m+1} e^{\frac{m\pi}{2\epsilon}(\eta-\beta)} \sin m\theta_1 \int_{\eta-\beta}^{\infty} e^{-(n+\frac{m\pi}{2\epsilon})\lambda} d\lambda$$

$$=2\sum_{m=1}^{\infty}(-1)^{m+1} \frac{e^{-n(\eta-\beta)}}{n+\frac{m\pi}{2\epsilon}} \sin m\theta_1 \,;$$

$$\int_{0}^{\eta-\beta} \frac{\sin\theta_1 e^{-n\lambda}\, d\lambda}{\cos\theta_1 + \cosh\frac{\pi}{2\epsilon}(\lambda-\eta+\beta)}$$

$$=2\sum_{m=1}^{\infty}(-1)^{m+1} e^{-\frac{m\pi}{2\epsilon}(\eta-\beta)} \sin m\theta_1 \int_{0}^{\eta-\beta} e^{\left(-n+\frac{m\pi}{2\epsilon}\right)\lambda} d\lambda$$

$$=2\sum_{m=1}^{\infty}(-1)^{m+1} \frac{e^{-n(\eta-\beta)} - e^{-\frac{m\pi}{2\epsilon}(\eta-\beta)}}{-n+\frac{m\pi}{2\epsilon}} \sin m\theta_1 .$$

$$\therefore \int_{0}^{\infty} \frac{\sin\theta_1 e^{-n\lambda}\, d\lambda}{\cos\theta_1 + \cosh\frac{m\pi}{2\epsilon}(\lambda-\eta+\beta)}$$

$$=2e^{-n(\eta-\beta)}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{4m\pi\epsilon}{m^2\pi^2-4n^2\epsilon^2}\sin m\theta_1$$

$$-4\epsilon\sum_{m=1}^{\infty}(-1)^{m+1}\frac{e^{-\frac{m\pi}{2\epsilon}(\eta-\beta)}}{m\pi-2n\epsilon}\sin m\theta_1 .$$

But we have, when ξ lies between a_1 and a_2

$$\frac{\sin n(\xi-a_1)}{\sin 2n\epsilon} = 2\sum_{m=1}^{\infty}(-1)^{m+1}\frac{m\pi}{m^2\pi^2-4n^2\epsilon^2}\sin\frac{m\pi}{2\epsilon}(\xi-a_1)\,;$$

$$\therefore \int_0^\infty \frac{\sin\frac{\pi}{2\epsilon}(\xi-a_1)e^{-n\lambda}\,d\lambda}{\cos\frac{\pi}{2\epsilon}(\xi-a_1)+\cosh\frac{\pi}{2\epsilon}(\lambda-\eta+\beta)}$$

$$=\frac{4\epsilon e^{-n(\eta-\beta)}\sin n(\xi-a_1)}{\sin 2n\epsilon}$$

$$-4\epsilon\sum_{m=1}^{\infty}(-1)^{m+1}\frac{e^{-\frac{m\pi}{2\epsilon}(\eta-\beta)}}{m\pi-2n\epsilon}\sin\frac{m\pi}{2\epsilon}(\xi-a_1)$$

$$\therefore I_1=\frac{\pi}{2}\Bigg[\frac{\sin n(\xi-a_1)}{\sin 2n\epsilon}e^{-n(\eta-\beta)}$$

$$-\sum_{m=1}^{\infty}(-1)^{m+1}\frac{2m\pi}{m^2\pi^2-4n^2\epsilon^2}\sin\frac{m\pi}{2\epsilon}(\xi-a_1)e^{-\frac{m\pi}{2\epsilon}(\eta-\beta)}\Bigg].$$

$$\therefore \sum_{n=1}^{\infty}\Bigg\{R_n\int_0^\infty e^{-n\lambda}\,d\lambda\int_0^\infty\frac{\sinh a(\xi-a_1)}{\sinh 2a\epsilon}\sin a\lambda\sin a(\eta-\beta)da$$

$$+S_n\int_0^\infty e^{-n\lambda}\,d\lambda\int_0^\infty\frac{\sinh a(\xi-a_2)}{\sinh 2a\epsilon}\sin a\lambda\sin a(\eta-\beta)da\Bigg\}$$

$$=2\sum_{n=1}^{\infty}(-1)^{n+1}e^{-n\beta}$$

$$\times\coth\beta\,\frac{\cos na_2\sin n(\xi-a_1)-\cos na_1\sin n(\xi-a_2)}{\sin 2n\epsilon}e^{-n(\eta-\beta)}$$

$$+2\sum_{n=1}^{\infty}(-1)^{n+1}e^{-n\beta}\coth\beta\sum_{m=1}^{\infty}(-1)^{m}\frac{2m\pi}{m^2\pi^2-4n^2\epsilon^2}$$

$$\times e^{-\frac{m\pi}{2\epsilon}(\eta-\beta)}\Big[\cos na_2\sin\frac{m\pi}{2\epsilon}(\xi-a_1)-\cos na_1\sin\frac{m\pi}{2\epsilon}(\xi-a_2)\Big]$$

$$=2\coth\beta\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos na_2\sin n(\xi-a_1)-\cos na_1\sin n(\xi-a_2)}{\sin 2n\epsilon}e^{-n\eta}$$

$$+4\sum_{m=1}^{\infty}\left\{A_m\sin\frac{m\pi}{2\epsilon}(\xi-a_1)+B_m\sin\frac{m\pi}{2\epsilon}(\xi-a_2)\right\}e^{-\frac{m\pi}{2\epsilon}(\eta-\beta)} \quad (6)$$

where

$$A_m=\sum_{n=1}^{\infty}(-1)^{m+n+1}\frac{m\pi}{m^2\pi^2-4n^2\epsilon^2}e^{-\eta\beta}\coth\beta\cos na_2 \quad \ldots \quad (7)$$

$$B_m=\sum_{n=1}^{\infty}(-1)^{m+n}\frac{m\pi}{m^2\pi^2-4n^2\epsilon^2}e^{-\eta\beta}\coth\beta\cos na_1.$$

But $$\cos na_2\sin n(\xi-a_1)-\cos na_1\sin n(\xi-a_2)$$

$$=\cos n\xi\sin n(a_2-a_1);$$

therefore

$$\psi=2c^2\omega\sum_{n=1}^{\infty}(-1)^n\frac{\sin na_2\cot a_2\sin n(\xi-a_1)+\sin na_1\cot a_1\sin n(a_2-\xi)}{\sin n(a_2-a_1)}e^{-n\eta}$$

$$+4c^2\omega\sum_{m=1}^{\infty}\left\{A_m\sin\frac{m\pi}{2\epsilon}(\xi-a_1)+B_m\sin\frac{m\pi}{2\epsilon}(\xi-a_2)\right\}e^{-\frac{m\pi}{2\epsilon}(\eta-\beta)}$$

$$+4c^2\omega\sum_{m=0}^{\infty}C_m\cos\frac{(2m+1)\pi}{2\epsilon}(\xi-\gamma)\,e^{-\frac{(2m+1)\pi}{2\epsilon}(\eta-\beta)} \quad \ldots \quad (8)$$

where

$$C_m=\sum_{n=1}^{\infty}(-1)^{m+n+1}\frac{(2m+1)\pi}{(2m+1)^2\pi^2-4n^2\epsilon^2}$$

$$\times(\sin na_2\cot a_2+\sin na_1\cot a_1)\,e^{-n\beta}; \quad \ldots \quad (9)$$

therefore

$$\phi=2c^2\omega\sum_{n=1}^{\infty}(-1)^n\frac{\sin n\alpha_2\cot\alpha_2\cos n(\xi-\alpha_1)-\sin n\alpha_1\cot\alpha_1\cos n(\xi-\alpha_2)}{\sin n(\alpha_2-\alpha_1)}e^{-n\eta}$$

$$+4c^2\omega\sum_{m=1}^{\infty}\{A_m\cos\frac{m\pi}{2\epsilon}(\xi-\alpha_1)+B_m\cos\frac{m\pi}{2\epsilon}(\xi-\alpha_2)\}e^{-\frac{m\pi}{2\epsilon}(\eta-\beta)}$$

$$-4c^2\omega\sum_{m=0}^{\infty}C_m\sin\frac{(2m+1)\pi}{2\epsilon}(\xi-\gamma)\,e^{-\frac{(2m+1)\pi}{2\epsilon}(\eta-\beta)}\quad\ldots\quad(10)$$

4. The following particular cases can be deduced from the case of three circles.

(*1*) *Two orthogonal circles*

Put $\alpha_2=\alpha$ and $\alpha_1=-(\pi-\alpha)$

$\therefore$ we get an area bounded by two orthogonal circles.

$$2\epsilon=\pi \text{ and } 2\gamma=2\alpha-\pi$$

then after a little simplification we have

$$\psi=2c^2\omega\cot\alpha\sum_{n=1}^{\infty}(-1)^n e^{-n\eta}\sin n\xi$$

$$+4c^2\omega\cot\sum_{m=a}^{\infty}M\sin m(\xi-\alpha)\,e^{-m(\eta-\beta)}$$

$$+4c^2\omega\sum_{m=1}^{\infty}N\sin(2m+1)(\xi-\alpha)\,e^{-(2m+1)(\eta-\beta)}$$

where

$$M=\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{m}{m^2-n^2}e^{-n\beta}\coth\beta\cos n\alpha\left[(-1)^{n+1}+(-1)^n\right]$$

and

$$N=\frac{2}{\pi}\sum_{p=1}^{\infty}\frac{(2m+1)}{(2m+1)^2-4p^2}\cot\alpha\sin 2p\alpha\cdot e^{-2p\beta}$$

The expression for ϕ can be written down.

(2) *Semi-lune*

If we put $\beta=0$, we get an area bounded by two circles and a straight line cutting them at right angles.

The boundary condition for $\beta=0$ is

$$\psi=\underset{\eta=0}{\mathrm{Lt}}\; c\omega y\coth\eta$$

$$=\underset{\eta=0}{\mathrm{Lt}}\;\frac{c+2c\sum_{1}^{\infty}(-1)^n e^{-n\eta}\cos n\xi}{\tanh\eta}$$

$$=\underset{\eta=0}{\mathrm{Lt}}\; c\omega\frac{2c\sum_{1}^{\infty}(-1)^{n+1}ne^{-n\eta}\cos n\xi}{\operatorname{sech}^2\eta}$$

$$=2c^2\omega\sum(-1)^{n+1}n\cos n\xi.$$

The boundary condition when $\eta=\beta$ is

$$\psi=2c^2\omega\sum(-1)^n e^{-n\beta}\coth\beta\cos n\xi$$

Replacing $e^{-n\beta}\coth\beta$ by $-n$ we have

$$\psi=2c^2\omega\sum_{n=1}^{\infty}(-1)^n\frac{\sin n\alpha_2\cot\alpha_2\sin n(\xi-\alpha_1)+\sin n\alpha_1\cot\alpha_1\sin n(\alpha_2-\xi)}{\sin n(\alpha_2-\alpha_1)}e^{-n\eta}$$

$$+4c^2\omega\sum_{m=1}^{\infty}\left\{A_m\sin\frac{m\pi}{2\epsilon}(\xi-\alpha_1)+B_m\sin\frac{m\pi}{2\epsilon}(\xi-\alpha_2)\right\}e^{-\frac{m\pi}{2\epsilon}\eta}$$

$$+4c^2\omega\sum_{m=0}^{\infty}C_m\cos\frac{(2m+1)\pi}{2\epsilon}(\xi-\gamma)\,e^{-\frac{(2m+1)\pi}{2\epsilon}\eta}$$

where

$$A_m = \sum_{n=1}^{\infty} (-1)^{m+n} \frac{mn\pi}{m^2\pi^2 - 4n^2\epsilon^2} \cos n\alpha_2$$

$$B_m = \sum_{n=1}^{\infty} (-1)^{m+n+1} \frac{mn\pi}{m^2\pi^2 - 4n^2\epsilon^2} \cos n\alpha_1$$

$$C_m = \sum_{n=1}^{\infty} (-1)^{m+n+1} \frac{(2m+1)\pi}{(2m+1)^2\pi^2 - 4n^2\epsilon^2} (\sin n\alpha_2 \cot \alpha_2 + \sin n\alpha_1 \cot \alpha_1)$$

(3) *Area bounded by a circle and two chords at right angles.*

Put $\beta=0,\ \alpha_1=0,\ \alpha_2=\alpha$

$\therefore\ 2\gamma=\alpha,\ 2\epsilon=\alpha$

$$\psi = 2c^2\omega \sum_{n=1}^{\infty} (-1)^n \frac{\sin n\alpha \cot \alpha \sin n\xi + n \sin n(\alpha-\xi)}{\sin n\alpha} e^{-n\eta}$$

$$+4c^2\omega \sum_{m=1}^{\infty} L_m \sin \frac{m\pi\xi}{\alpha} e^{-\frac{m\pi\eta}{\alpha}}$$

$$+4c^2\omega \sum_{m=0}^{\infty} D_m \sin \frac{(2m+1)\pi\xi}{\alpha} e^{-\frac{(2m+1)\pi\eta}{\alpha}}$$

where

$$L_m = \sum_{n=1}^{\infty} (-1)^n \frac{mn\pi}{m^2\pi^2 - n^2\alpha^2} [(-1)^m \cos n\alpha - 1]$$

$$D_m = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2m+1)\pi}{(2m+1)^2\pi^2 - n^2\alpha^2} (\sin n\alpha \cot \alpha + n).$$

Review

It is a striking evidence of the intellectual virility of the Germans that even under their present politically depressing conditions they continue to make solid and considerable contributions to the creation and diffusion of mathematical knowledge. Work in connection with the Encyclopædia of Mathematical Sciences initiated by Professor Felix Klein in the nineties of the last century is being continued almost without any interruption with the result that during the last five years numerous parts of the Encyclopædia have been brought out. A new journal of research, *viz.*, the *Mathematische Zeitschrift*, came into existence after the armistice and has proved itself to be very useful for the cause of mathematical research. A number of new series of mathematical books have been planned in the last five years. Of one of such series, *viz.*, "Die Grundlehren der Mathematischen Wissenschaften," ably edited by Professor Courant of Göttingen with the co-operation of Professor Blaschke of Hamburg and Professors Runge and Born of Göttingen, we propose to write at some length in the following lines:

"The series is intended to consist of introductory text-books, each independent of any other, for the most important branches of Pure and Applied Mathematics." The modern requirements of mathematical rigour are kept in view and the needs of the physicists and engineers are not lost sight of. In each case, the text is illustrated by a number of problems and examples.

Up to this time the following seven volumes have been published:

Blaschke's "Lectures on Differential Geometry and geometrical foundations of Einstein's theory of relativity" in 2 Vols.

Knopp's "Theory and application of infinite series."

Adolf Hurwitz's "Lectures on the general theory of functions and elliptic functions" brought out and completed by Prof. Courant.

Madelung's "Mathematical means for physicists."

Speiser's "The Theory of groups of finite order."

Bieberbach's "Theory of differential equations."

In the present issue of the Bulletin we propose to review the book of Bieberbach reserving the review ofthe other books of the series to subsequent issues.

Professor Bieberbach's book is specially remarkable because of its being originally conceived and also because of its being thoroughly

up to date in all those matters which it deals with. The author's conception of an ideal text-book on differential equations is a book which should enable the reader to study *any* original paper on the subject with understanding and without feeling that he is studying something quite strange to him. Professor Bieberbach declares this ideal to be impossible of realization and has therefore contented himself with a wise choice of topics some of which, although of the greatest importance, have up to now found no adequate place in any text book.

The author divides the book into four parts. The first part deals with the ordinary differential equations of the first order. The second with those of the second order, the third with partial differential equations of the first order and the last with partial differential equations of the second order.

The first part is divided into four chapters which respectively treat of elementary methods of integration, the method of successive approximations, the discussion of the course of the integral-curves and the differential equations of the first order in the complex domain.

The second part is also divided into four chapters which are respectively headed, the existence of solutions, elementary methods of integration, discussion of the course of the integral-curves and linear differential equations of the second order in the complex domain.

The third part has only one chapter and the fourth is divided into four chapters which bear the titles, general, hyperbolic differential equations, elliptic differential equations and parabolic differential equations.

On the second page the author makes the following striking pronouncement: "It is, however, always the object of the theory to aim at the determination of the properties of those functions which satisfy a given differential equation or a given system of differential equations. The smallest requirement is, to find an explicit expression for these functions. It is more important to deduce how the functional character, therefore, for example, the course of the curve-image, of the functions giving the solution, depends on the properties of the differential equations and can be determined by them." The first chapter of the first part gives an interesting and instructive exposition of the most elementary topics, including the standard forms. The second, third and fourth chapters deal with matters which have found practically no mention in even the most recent edition of Professor Forsyth's book so well-known to students in India. Professor Bieberbach begins the second chapter with a careful statement of the existence—theorem which may be usefully reproduced here:

"In the differential equation

$$\frac{dy}{dx}=f(x, y),$$

for a given convex region B of the x, y plane, let, $f(x, y)$ be continuous and satisfy for each point-pair (x, y_1) and (x, y_2) in that region the condition of Lipschitz

$$| f(x, y_1)-f(x, y_2) | \leq M | y_1-y_2 | ,$$

where M is a suitable positive number independent of x and of y_1 and y_2. Further, in B let

$$| f(x, y) | < M.$$

Further let a and b be two positive numbers which satisfy the conditions

$$b > aM$$

and for which the rectangle

$$| x-x_0 | < a,$$

$$| y-y_0 | < b$$

belongs to the region B.

Then there is exactly one function

$$y=\phi(x),$$

continuous along with its first differential co-efficient, which satisfies the differential equation, for which, therefore, in

$$| x-y_0 | < a$$

$$\phi'(x)=f(x, \phi(x))$$

holds and which at the same time passes through the point (x_0, y_0) for which therefore

$$\phi(x_0)=y_0."$$

In the third chapter, a singular point is defined as a point where one or more of the suppositions made in the afore-mentioned theorem are not true. With a wealth of simple cases, *e.g.*, $y'=+\sqrt{| y |}$, it is brought home to the reader that although the mere continuity of $f(x, y)$ ensures the existence of at least one solution through every

point, a number of solutions may pass through that point. A classification of the various types of singular points closes the preliminary exposition after which general theorems on the course of integral curves in the real region are given and then a careful and lucid exposition of the investigations of Poincare, Bendixson and Perron. The third chapter closes with an interesting discussion of singular solutions. The fourth chapter opens with the definition of fixed and moveable singularities, proceeds with the treatment of differential equations with single-valued integrals and ends with the discussion of the behaviour of the integrals in the neighbourhood of singular points.

A detailed review of the remaining parts is not desirable. Suffice it to mention that the treatment of the differential equations of the second order includes not only the salient features of the first part but also an interesting exposition of the relation between those differential equations and the theory of integral equations, that the discussion of partial differential equations is not only simple but rigorous and that most of the latest researches on the subject of the nature of the solutions of partial differential equations of the second order find mention in the last part of the book.

Professor Bieberbach has laid the mathematical public under a deep debt of gratitude by placing before it a book which is full of suggestive results and is a marvel of lucidity and rigour.

GANESH PRASAD

Sir Asutosh Mookerjee

Life Sketch

Sir Asutosh Mookerjee was born in Calcutta on June 29, 1864. His father Dr. Gangaprasad Mookerjee was an eminent physician and a much esteemed citizen of Calcutta. After finishing the preliminary course in a vernacular school (1869-72) Asutosh was taken in hand by his father under whose direct supervision he prosecuted his studies till 1875 when he was admitted into the South Suburban School and matriculated in 1879 at the age of 15 standing second in the list of the successful candidates. His undergraduate career in the Presidency College (1880-84) was one of uniform brilliance and in 1884 he topped the list in the B.A. Examination. He took the degree of M.A. in Mathematics next year and in physical Science in 1886. The same year he was awarded the Premchand Roychand Studentship and admitted as a Fellow of the Royal Society of Edinburgh. He completed his law lectures in the City College and passed B.L. in 1888. The same year he was enrolled as a Vakil of the High Court, having at the same time completed the period of articleship under the late Sir Rash Behary Ghosh. He had been a Fellow of the Calcutta University since 1889. The Doctorate of law was conferred on him in 1894 and his "Law of Perpetuity" embodying the lectures delivered as a Tagore Law Lecturer is no less authoritative, though not so well-known, than the Law of Mortgage by his legal *Guru* Rash Behary.

He entered the Bengal Legislative Council in 1899 as the representative of the University and was re-elected, two years later. He represented Bengal in the Indian Universities' Commission appointed by Lord Curzon and took his seat in the Provincial Legislative Council for the third time in 1903 as the representative of Calcutta Corporation. The same year the Bengal Council sent him to the Imperial Legislative Council as its representative. In 1904 he became a judge of the Calcutta High Court.

He was made Vice-Chancellor of the Calcutta University in 1906 in succession to Dr. Gooroodas Banerjee and held that honorary but highly responsible post until 1914. In 1907 he was elected President of the Asiatic Society of Bengal, an office to which he was subsequently repeatedly elected. In 1908 he founded the Calcutta Mathematical

Society and was its first President at the time of his death. He accepted the Vice-Chancellorship again in 1921 for two years at the special request of Lord Chelmsford and Lord Ronaldshay. He resigned his post as a Judge of the High Court in December, 1923. On the 26th of June, 1924, he died at Patna.

What Sir Asutosh Mookerjee did for the Calcutta University may be gauged by the various tributes of respect paid to his memory after his death. Of these, the following resolution by the Syndicate of the Calcutta University may be quoted :—

We, the members of the Syndicate, at a special meeting convened for the purpose, place on record an expression of our profound grief at the death of our revered colleague, Sir Asutosh Mookerjee. As Vice-Chancellor or as an ordinary member of the Syndicate he had been intimately associated with its work since 1889. For thirty-five years he placed his outstanding intellectual powers and his unrivalled energy ungrudgingly at the service of his colleagues, thereby enabling them to carry out a task which year by year became more difficult, laborious and exacting. The remarkable developments in the work of the University during the last two decades which it was our privilege as the representatives of the Senate to direct, were largly the product not only of his constructive genius but of the selfless, incessant and devoted toil, which he brought to his task as a member of our body. The personal and private sorrow which we each individually feel at the loss of our distinguished colleague is intensified by our keen sense of the irreparable injury to our work which will be caused by the absence of his indefatiguable energy, his directive skill and his unique knowledge and experience. In paying our sorrowful tribute of respect to the friend, colleague, and leader whom we have lost, and in placing on record our profound admiration for the services rendered to the cause of education by the work which he accomplished as a member of our body, we express the hope that the memory of his devoted labours may inspire those of us who remain, and those who follow us, to imitate his great example, and dedicate all the powers which they possess to the service of their University and to the achievement of that object for which he lived, the advancement of learning amongst the people of his motherland.

Sir Asutosh's contributions to the stock of mathematical knowledge consists of nearly twenty papers published in the *Journal of the Asiatic Society of Bengal*, the *Messenger of Mathematics* and the *Quarterly*

Journal of Mathematics, during the years 1880-1890. Sir Asutosh's boldness of vision and independence of thought—qualities absolutely necessary for success as a mathematical investigator—showed themselves very early when, as a student of the first year class of the Presidency College, he wrote his first paper in 1880. This paper contains a new proof of the 25th proposition of the first book of Euclid, and it appeared in vol. 10 of the *Messenger of Mathematics* of Cambridge in 1881. Whilst still an undergraduate Sir Asutosh wrote a paper on some extensions of a theorem of Salmon's and this paper appeared in vol. 13 of the *Messenger of Mathematics* in 1884. In October, 1884, he took a step which brought him to the threshhold of what might have been a career full of high achievements as a mathematical investigator : whilst a student of the fifth year class he gave a new method for solving Euler's equation based on the properties of the ellipse. Sir Asutosh was a careful student ; and he was so confident of his own abilities and intellectual powers that as soon as he came across any problem which challanged his intellect, he would at once attack it. If, therefore, he had studied with the same care a book on elliptic functions more modern in its contents than Cayley's book, *e.g.*, Briot and Bouquet's " Théorie des fonctions elliptiques " which had appeared as early as 1859, Sir Asutosh would have certainly made many important contributions to the theory of functions in general and the theory of elliptic functions in particular. But unfortunately, like many Indians in the eighties, he thought that everything worth knowing about a mathematical subject could be found in English books. As a matter of fact, English mathematicians were themselves very late in picking up the new theories that had come into existence and been developed on the continent. Sir Asutosh had no guidance in his research work ; there was no one in India in the eighties of the last century who could be worthy of being his guide. In 1887, Sir Asutosh took up the study of Monge's differential equation for the general conic. Boole had stated in his book on differential equations that the differential equation of Monge has not been geometrically interpreted. Sir Asutosh undertook to obtain the true geometrical interpretation and criticised the interpretations given by Professor Sylvester and Lt. Colonel Cunningham which were respectively the following :—

" That the differential equation of a conic is satisfied at the sextactic points of any given curve " and

"That the eccentricity of the osculating conic of a given conic is constant all round the latter."

Sir Asutosh devoted more time to this problem than was necessary, considering the fact that its solution was even then a matter of no great importance and is now of interest as a mere curiosity. Sir Asutosh published altogether four papers on Monge's differential equation and gave as the true geometrical interpretation the following :—

"The radius of curvature of the aberrancy curve (discussed by Transon) vanishes at every point of every conic."

Among the other publications of Sir Asutosh may be mentioned two papers on isogonal trajectories, two papers on Hydrodynamics, a paper on an integral of Poisson and a paper on the determination of certain mean values by means of elliptic functions. Sir Asutosh's contributions to mathematical knowledge were due to his unaided efforts while he was only a college student. After Bhaskara, he was the first Indian to enter into the field of mathematical research as distinguished from astronomical research, and did much which was truly original.

A list of the papers of Sir Asutosh Mookerjee is appended to this obituary.

GANESH PRASAD

MATHEMATICAL PAPERS

BY

SIR ASUTOSH MOOKERJEE

(*i*) "On a Geometrical Theorem" (*Messenger of Mathematics*, Vol. 10, p. 122).

(*ii*) "Extensions of a Theorem of Salmon's" (*Messenger of Mathematics*, Vol. 13, p. 157).

(*iii*) Mathematical Notes (Reprints from the *Educational Times* of London).

(*iv*) "Note on Elliptic Functions" which has been referred to in Enneper's *Elliptische Funktionen* (*Quarterly Journal of Pure and Applied Mathematics*, Vol. 21, p. 212).

(*v*) "On the Differential Equation of a Trajectory" which has been referred to in Forsyth's *Differential Equations* (*Journal of the Asiatic Society of Bengal*, Vol. 56, Part II, p. 116).

(*vi*) "On Monge's Differential Equation to all Conics" (*ibid*, p. 134).

(*vii*) "Memoir on Plane Analytical Geometry" (*ibid*, p. 188).

(*viii*) "A General Theorem on the Differential Equations of the Trajectories" (*ibid*, Vol. 57, Part II, p. 72).

(*ix*) "On Poisson's Integral" (*ibid*, p. 100).

(*x*) "On the Differential Equation of all Parabolas" (*ibid*, p. 316).

(*xi*) "The Geometric interpretation of Monge's Differential Equation ~~to~~ to all Conics" which has been cited in Edward's *Differential Calculus* (*ibid*, Vol. 58, Part II, p. 181).

(*xii*) "Some Applications of Elliptic Functions to Problems of Mean Values, Parts I and II" (*ibid*, pp. 199 and 213).

(*xiii*) "On Clebsch's Transformation of Hydrokinetic Equations and Note on Stokes's Theorem of Hydrokinetic Circulation" (*ibid*, Vol. 59, pp. 56 and 59).

(*xiv*) On a Curve of Aberrancy (*ibid*, Vol. 59, p. 61).

(*xv*) "Remarks on Monge's Equations to all Conics" (*ibid*, 1888).

(*xvi*) "On Some Definite Integrals."

(*xvii*) "On an Application of Differential Equations to the Theory of Plane Cubics."

(*xviii*) "Researches on the Number of Normals common to Two Surfaces, Two Curves, or a Curve and a Surface."

(*xix*) "Application of Gauss's Theory of Curvature to the Evaluation of Double Integrals."

On the fundamental theorem of the Integral Calculus

By

Ganesh Prasad

According to the fundamental theorem of the Integral Calculus, the integral function

$$F(x) \equiv \int f(x)dx$$

has a differential coefficient $f(x)$ at any point of continuity of $f(x)$. But the usual proof to be found in books on the theory of functions of a real variable fails altogether when we consider the question of the existence of the differential coefficient of $F(x)$ at a point where $f(x)$ has a discontinuity of the *second* kind.

The object of the present paper is to investigate the conditions, which must be satisfied by $f(x)$ or the types of functions to which $f(x)$ must belong, in order that $F(x)$ should have a differential coefficient at a point of discontinuity of the second kind of $f(x)$. The results obtained by me are believed to be all new.

For the sake of simplicity and fixity of ideas, I take the point of discontinuity to be $x=0$ and represent the integral function as

$$\int_0^x f(x)dx.$$

Throughout the paper $\psi(x)$ denotes a function which is monotone in the neighbourhood of $x=0$ and which tends to infinity as x tends to 0.

I. $f(x)=\cos\psi(x)$

§ 1. Let $f(x)=\cos\psi(x)$.

Then, at $x=0$, $F(x)$ has a differential coefficient equal to zero when

$$\psi(x) \succ \log\left(\frac{1}{x^2}\right),$$

and $F(x)$ has no differential coefficient when

$$\psi(x) \precsim \log\left(\frac{1}{x^2}\right).$$

Proof:—

(*a*) *Case*: $$\psi(x) \succ \log\left(\frac{1}{x^2}\right).$$

Since

$$\frac{d}{dx}\left\{\frac{1}{\psi'}\sin\psi\right\} = \cos\psi - \frac{\psi''}{(\psi')^2}\sin\psi,$$

integrating both the sides of the above equality, we have

$$\frac{1}{\psi'}\sin\psi = \int_0^x \cos\psi\, dx - \int_0^x \frac{\psi'}{(\psi')^2}\sin\psi\, dx,$$

i.e.,

$$\int_0^x \cos\psi\, dx = \frac{\sin\psi}{\psi'} + \int_0^x \frac{\psi''}{(\psi')^2}\sin\psi\, dx.$$

Now consider

$$\lim_{x=+0} \frac{F(x)}{x}.$$

This equals

$$\lim_{x=+0} \frac{\sin\psi}{x\psi'} + \lim_{x=+0} \frac{1}{x}\int_0^x \frac{\psi''}{(\psi')^2}\sin\psi\, dx. \qquad \ldots \quad (1)$$

But, since

$$\psi \succ \log\frac{1}{x},$$

$$\psi' \succ \frac{1}{x},$$

i.e., $$x\psi' \succ 1;$$

also $$\frac{\psi''}{(\psi')^2} \prec 1$$

and consequently

$$\frac{\psi''}{(\psi')^2}\ \sin\psi$$

is continuous at $x=0$. Therefore both the limits in (1) exist and equal zero. Therefore

$$\lim_{x=+0}\ \frac{F(x)}{x}$$

exists and equals zero.

Similarly it can be proved that

$$\lim_{x=-0}\ \frac{F(x)}{x}$$

exists and equals zero.

Hence $F'(0)$ exists and equals zero.

(*b*) *Case*: $$\psi(x) \prec \log\left(\frac{1}{x^2}\right).$$

Integrating by parts,

$$\int_0^x \cos\psi\, dx = x\cos\psi + \int_0^x x\psi' \sin\psi\, dx.$$

Now,

$$\psi' \prec \frac{1}{x},\quad i.e.,\quad x\psi' \quad 1.$$

Therefore $x\psi' \sin\psi$ is continuous at $x=0$ and, consequently,

$$\int_0^x x\psi' \sin\psi\, dx$$

has a differential coefficient zero at $x=0$. Therefore, at $x=0$, $F(x)$ has no differential coefficient, for $x\cos\psi$ has no differential coefficient there.

(*c*) *Case*: $$\psi(x) \backsim \log\left(\frac{1}{x^2}\right).$$

Let $$\psi(x) = a + \log\frac{1}{x},$$

where a is a constant, and x is positive. Then,

$$F(x)=\cos a.\int_0^x \cos\log\frac{1}{x}\,dx-\sin a\int_0^x \sin\log\frac{1}{x}\,dx$$

$$=x\cos a.\ \frac{\cos\log\frac{1}{x}-\sin\log\frac{1}{x}}{2}-x\sin a.\ \frac{\cos\log\frac{1}{x}+\sin\log\frac{1}{x}}{2}$$

$$=x\cos\log\frac{1}{x}.\ \frac{\cos a-\sin a}{2}-x\sin\log\frac{1}{x}.\ \frac{\cos a+\sin a}{2}$$

$$=\frac{x}{\sqrt{2}}\cos\left\{a+\log\frac{1}{x}+\frac{\pi}{4}\right\}$$

$$=\frac{x}{\sqrt{2}}\cos\left(\psi+\frac{\pi}{4}\right).$$

Therefore, whatever a may be, $F'(0)$ is non-existent.

Generally, let

$$\psi(x)=\frac{1}{2}\left\{\sigma(x)+1\right\}\log\frac{1}{x^2},$$

where $\sigma(x)\prec 1$.

Then, proceeding as in (b), we have

$$\int_0^x \cos\psi\,dx=x\cos\psi+\int_0^x x\psi'\sin\psi\,dx$$

$$=x\cos\psi+\int_0^x\left\{x\sigma'\log\frac{1}{x}-(1+\sigma)\right\}\sin\psi\,dx$$

$$=x\cos\psi-\int_0^x\sin\psi\,dx+G(x),$$

where G, being the integral function of a continuous integrand which tends to 0 with x, has a differential coefficient zero at $x=0$.

Therefore in the limit we may take

$$\int_0^x \cos\psi\,dx = x\cos\psi - \int_0^x \sin\psi\,dx.$$

Similarly

$$\int_0^x \sin\psi\,dx = x\sin\psi + \int_0^x \cos\psi\,dx.$$

Therefore, as x tends to zero,

$$\int_0^x \cos\psi\,dx$$

behaves as

$$x.\ \frac{\cos\psi - \sin\psi}{2},$$

i.e., as

$$\frac{x}{\sqrt{2}}\cos\left(\psi + \frac{\pi}{4}\right).$$

Hence $F'(0)$ is non-existent.

II. $f(x) = \chi(x)\cos\psi(x)$, χ *being monotone and limited*

§ 2. If $f(x)=\chi(x)\cos\psi(x)$, where $\chi(x)$ is monotone and of the form $A+\chi_1(x)$, A being a constant, different from zero, and $\chi_1 \prec 1$, then $F'(0)$ exists or not according as

$$\psi \succ \log\frac{1}{x^2} \quad \text{or} \quad \psi \preceq \log\frac{1}{x^2}.$$

Example.

$$\int_0^x \left(1+\sqrt{x}\right)\cos\frac{1}{x}\,dx$$

has a differential coefficient zero at $x=0$.

III. $f(x)=\chi(x)\cos\psi(x)$, χ *being limited but not monotone*

§ 3. Let $f(x)=\chi(x)\cos\psi(x)$, where $\chi(x)$ is *not* monotone but is of the form $A+\chi_1(x)$, A being a constant different from zero and χ_1 being equal to $B\cos\psi_1(x)$, where B is a constant different from zero and $\psi_1 \succ 1$.

Then $F'(0)$ exists or not according as

$$\psi \succ \log\frac{1}{x^2}, \quad \text{or } \psi \precsim \log\frac{1}{x^2},$$

ψ_1 *not* being $\backsim \psi$.

Proof:—

$$F(x)=\int_0^x (A+B\cos\psi_1)\cos\psi\, dx$$

$$=A\int_0^x \cos\psi\, dx+\frac{B}{2}\int_0^x \cos(\psi+\psi_1)dx+\frac{B}{2}\int_0^x \cos(\psi-\psi_1)dx.... \quad (2)$$

Now, when $\psi \succ \log\frac{1}{x^2}$,

$$\psi+\psi_1 \succ \log\frac{1}{x^2}, \text{ and also } \psi-\psi_1 \succ \log\frac{1}{x^2}.$$

Therefore by § 1, each of the three integral functions in the equation (2) to whose sum $F(x)$ is equal has a differential coefficient zero at $x=0$; consequently $F'(0)$ exists and is zero.

When $\psi \precsim \log\frac{1}{x^2}$,

then two cases arise; either

$$\psi_1 \succ \log\frac{1}{x^2}, \quad \text{or } \psi_1 \precsim \log\frac{1}{x^2}.$$

In the first case the second and the third integral functions in (2) have each zero as their differential coefficients at $x=0$, and the first

integral function has no differential coefficient. Hence $F'(0)$ is non-existent. In the second case there are three possibilities according as

(i) $$\psi \prec \log\frac{1}{x^2} \text{ and } \psi_1 \prec \log\frac{1}{x^2},$$

(ii) $$\psi \prec \log\frac{1}{x^2} \text{ and } \psi_1 \backsim \log\frac{1}{x^2}$$

or

(iii) $$\psi \backsim \log\frac{1}{x^2} \text{ and } \psi_1 \prec \log\frac{1}{x^2}.$$

For (*i*), it follows from (*b*) of § 1, that $\frac{F(x)}{x}$ behaves as

$$A\cos\psi + \frac{B}{2}\cos(\psi+\psi_1) + \frac{B}{2}\cos(\psi-\psi_1),$$

and therefore $F'(0)$ is non-existent.

For (*ii*), it follows from (*b*) and (*c*) of § 1 that $\frac{F(x)}{x}$ behaves as

$$A\cos\psi + \frac{B}{2\sqrt{2}}\cos\left(\psi+\psi_1+\frac{\pi}{4}\right) + \frac{B}{2\sqrt{2}}\cos\left(\psi-\psi_1+\frac{\pi}{4}\right),$$

and therefore $F'(0)$ is non-existent.

For (*iii*), it follows from (*c*) of § 1 that $\frac{F(x)}{x}$ behaves as

$$\frac{A}{\sqrt{2}}\cos\left(\psi+\frac{\pi}{4}\right) + \frac{B}{2\sqrt{2}}\cos\left(\psi+\psi_1+\frac{\pi}{4}\right)$$

$$+ \frac{B}{2\sqrt{2}}\cos\left(\psi-\psi_1+\frac{\pi}{4}\right)$$

and therefore $F'(0)$ is non-existent.

Examples.

(1) $$\int_0^x \cos^2\left(\log\frac{1}{x^2}\right)\cos\frac{1}{x}\,dx$$

has a differential coefficient zero at $x=0$.

(2)
$$\int_0^x \cos^2\left(\log\frac{1}{x^2}\right)\sin\left\{\left(\log\frac{1}{x^2}\right)^{\frac{1}{2}}\right\}dx$$

has no differential coefficient at $x=0$.

§ 4. Consider now the case of $\psi_1 \asymp \psi$ and let $\psi_1=\psi(A+\sigma)$ where A is a constant different from zero and $\sigma \prec 1$. A number of cases arise.

(*i*) If $\psi \succ \log\frac{1}{x^2}$, the first two integral functions of equation (2) of the previous article have each zero as their differential coefficients at $x=0$; consequently $F'(0)$ exists or not according as

$$(\psi-\psi_1) \succ \log\frac{1}{x^2}, \text{ or } (\psi-\psi_1) \preceq \log\frac{1}{x^2},$$

which latter case can be possible only if $A=1$; provided that it is understood that $F'(0)$ always exists if $(\psi-\psi_1) \asymp 1$.

(*ii*) If $\psi \prec \log\frac{1}{x^2}$, then $(\psi+\psi_1) \prec \log\frac{1}{x^2}$ and either

$$1 \prec (\psi-\psi_1) \prec \log\frac{1}{x^2}, \quad \text{or} \quad (\psi-\psi_1) \asymp 1.$$

Therefore in this case it is easily proved by proceeding as in the preceding article that $F'(0)$ is non-existent.

(*iii*) If $\psi \asymp \log\frac{1}{x^2}$, then $(\psi+\psi_1) \asymp \log\frac{1}{x^2}$ and either

$$1 \prec (\psi-\psi_1) \preceq \log\frac{1}{x^2}, \quad \text{or} \quad (\psi-\psi_1) \asymp 1.$$

Therefore in this case it is easily proved by proceeding as in the preceding article that $F'(0)$ is non-existent.

Examples.

(1)
$$\int_0^x \left(1+\cos\frac{1}{x}\right)\cos\frac{1}{x}\,dx$$

has a differential coefficient at $x=0$ and is equal to $\frac{1}{2}$.

(2) $$\int_0^x \left(1+\cos\log\frac{1}{x^2}\right)\cos\log\frac{1}{x^2}\,dx$$

has no differential coefficient at $x=0$.

(3) $$\int_0^x \left\{1+\cos\sqrt{\left(\log\frac{1}{x^2}\right)}\right\}\cos\log\frac{1}{x^2}\,dx$$

has no differential coefficient at $x=0$.

(4) $$\int_0^x \cos^2\log\frac{1}{x^2}\cdot\sin\log\frac{1}{x^2}\,dx$$

has no differential coefficient at $x=0$.

IV. $f(x)=\chi(x)\cos\psi(x)$, $\chi(x)>1$

§ 5. Let $$f(x)=\chi(x)\cos\psi(x),$$

where $\chi(x)$ is monotone in the neighbourhood of $x=0$ and tends to infinity as x tends to zero. Then assuming that the improper integral

$$t\equiv\int_0^x \chi(x)dx$$

exists and $F(x)$ becomes

$$\int_0^t \cos\{\phi(t)\}dt,$$

$\phi(t)$ standing for $\psi(x)$, the criteria of § 1 are applicable.

Examples.

(1) $$\int_0^x \frac{1}{\sqrt{x}}\cos\frac{1}{\sqrt{x}}\,dx$$

has a differential coefficient at $x=0$.

(2)
$$\int_0^x \frac{1}{x^p} \cos \frac{1}{x^k} dx$$

has a differential coefficient at $x=0$, p being any positive proper fraction and k being any constant greater than zero.

(3)
$$\int_0^x \frac{1}{\sqrt{x}} \cos\left(\log \frac{1}{x^2}\right) dx$$

has no differential coefficient at $x=0$.

(4)
$$\int_0^x \frac{1}{x\left(\log \frac{1}{x^2}\right)^2} \cos\left(\log \frac{1}{x^2}\right) dx$$

has a differential coefficient zero at $x=0$.

V. *$f(x)=\chi(x) \cos \psi(x)$, $\chi(x)$ being neither limited nor monotone*

§ 6. If $f(x)=\chi(x) \cos \psi(x)$, where $\chi(x)$ is not monotone but makes an infinite number of fluctuations with indefinitely increasing amplitudes as x tends to zero, the procedures of the preceding articles cease to be applicable and in each case a special procedure is necessary.

Examples.

(1) Let
$$f(x) \equiv \left\{ 1+ \frac{1}{x} e^{\frac{1}{x}} \tan\left(e^{\frac{1}{x}}\right)\right\} \cos\left(e^{\frac{1}{x}}\right).$$

Then
$$\int_0^x f(x)dx = x \cos\left(e^{\frac{1}{x}}\right).$$

Therefore $F'(0)$ is non-existent.

(2) Let $$f(x)=\left\{1+\frac{1}{x}\tan\left(\frac{1}{x}\right)\right\}\cos\left(\frac{1}{x}\right).$$

Then $$\int_0^x f(x)dx=x\cos\frac{1}{x}.$$

Therefore $F'(0)$ is non-existent.

(3) Let $$f(x)=\left\{(1+k-p)x^{k-p}+\frac{k}{x^p}\tan\left(\frac{1}{x^k}\right)\right\}\cos\left(\frac{1}{x^k}\right),$$

where k and p are both greater than zero. Then

$$\int_0^x f(x)dx=x^{1+k-p}\cos\frac{1}{x^k}.$$

Therefore $F'(0)$ is existent or non-existent according as $k>p$ or not.

§ 7. If $f(x)$ makes an infinite number of fluctuations, not only in the neighbourhood of $x=0$ but also in the neighbourhood of any point $x=\omega_n$, $\{\omega_n\}$ being an enumerable and everywhere dense aggregate with 0 as a limiting point, then the procedures of the preceding articles cease to be applicable.

Example.

(1) Let $f(x)=\sum_1^\infty\frac{1}{2^n}\cos\frac{1}{x-\omega_n}$, where $\{\omega_n\}$ is the aggregate of rational numbers with 0 as a limiting point.

Now
$$\frac{d}{dx}\left[\sum_1^\infty(x-\omega_n)^2\frac{1}{2^n}\sin\frac{1}{x-\omega_n}\right]$$

$$=-\sum_1^\infty\frac{1}{2^n}\cos\frac{1}{x-\omega_n}+2\sum_1^\infty\frac{1}{2^n}(x-\omega_n)\sin\frac{1}{x-\omega_n},$$

term-by-term differentiation being permissible, as all the series concerned are uniformly convergent. Hence, integrating, we have

$$\left[-\sum_{1}^{\infty}(x-\omega_n)^2\frac{1}{2^n}\sin\frac{1}{x-\omega_n}\right]_0^x$$

$$+2\int_0^x\left\{\sum_{1}^{\infty}\frac{1}{2^n}(x-\omega_n)\sin\frac{1}{x-\omega_n}\right\}dx$$

$$=\int_0^x f(x)dx.$$

But obviously each of the two terms on the left side in the above equation has a differential coefficient for $x=0$. Therefore $F'(0)$ exists.

2

Tidal Oscillations on a Spheroid

By

B. M. Sen

The problem of tides on a globe was initiated by Laplace* and was further developed by Kelvin, Darwin, Airy and Hough. In his treatment of the tides on rotating globe, Laplace found the dynamical equations of tidal oscillations on a spheroid but passed off at once to the globe neglecting the eccentricity of the meridian section. The assumptions on which his treatment was based were the following:

(1) The motion is supposed small, so that the product and squares of the velocity are neglected.

(2) The pressure is the same as the hydrostatic pressure.

These are the usual assumptions of tidal oscillations.

(3) The free surface is an equipotential surface and the depth h of the liquid is supposed small, but arbitrary.

This implies that the surface of the spheroid is an equipotential surface or only slightly different from an equipotential surface.

(4) The ratio of the centrifugal force at the equator to the gravity, $\omega^2 a/g$, is supposed small.

(5) The eccentricity is neglected in the subsequent treatment.

(6) The attraction of the layer of liquid is neglected; this has been taken into account by subsequent writers.

In the following pages the problem of tidal oscillations on a spheroid rotating as well as non-rotating is dealt with. The eccentricity of the meridian section is not assumed small; the problem, moreover, differs from that of motion on a globe by the fact that h is prescribed by the necessary condition that the surface of the spheroid and also the free surface must be equipotential surfaces.

* Full references are given in Lamb's Hydrodynamics, Art. 213 *et seq.*

§ 1. Let the equation of the planetary spheroid be

$$\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1, \qquad \ldots \quad (1)$$

Introducing the spheroidal co-ordinates,

$$x=c\xi\zeta, \quad \omega=c\sqrt{1-\xi^2}\times\sqrt{1+\xi^2}, \qquad \ldots \quad (2)$$

the spheroid is given by the equation

$$\zeta=a, \text{ a constant.}$$

Then

$$\delta s^2=\frac{1}{h_1^2}\,\delta\xi^2+\frac{1}{h_2^2}\,\delta\zeta^2+\frac{1}{h_3^2}\,\delta\phi^2, \qquad \ldots \quad (3)$$

δs being an element of length in space where

$$h_1=\frac{1}{c}\left(\frac{1-\xi^2}{\xi^2+\zeta^2}\right)^{\frac{1}{2}}, \quad h_2=\frac{1}{c}\left(\frac{1+\zeta^2}{\xi^2+\zeta^2}\right)^{\frac{1}{2}},$$

$$h_3=\frac{1}{c}\,\frac{1}{(1-\xi^2)^{\frac{1}{2}}\,(1+\zeta^2)^{\frac{1}{2}}}. \qquad \ldots \quad (4)$$

§ 2. Consider first the case of no rotation. Let the depth of the liquid be h which is taken to be small. Neglecting the mutual attraction of the liquid particles, for equilibrium it is necessary that the surface of the spheroid be an equipotential. Now the potential of a solid homogeneous ellipsoid is given by the equation*

$$V=-\pi\,\rho abc\{\chi'-\alpha x^2-\beta y^2-\gamma z^2\}, \qquad \ldots \quad (5)$$

where $\displaystyle \chi'=\int_\lambda^\infty \frac{du}{Q}, \quad \alpha=\int_\lambda^\infty \frac{du}{(a^2+u)Q}$, etc.

and $\qquad Q^2=(a^2+u)(b^2+u)(c^2+u).$

In no case except that of the sphere, can the surface of the ellipsoid be an equipotential. We, therefore, make the assumption that the

* Routh, Analytical Statics, Vol. II, Art. 223, the sign of V having been reversed.

spheroid is heterogeneous and its equipotential surfaces are confocal spheroids. These are given by the relation ζ=const., and V is a function of ζ only.

§ 3. The equation of the meridian section is

$$\frac{\varpi^2}{\zeta^2}+\frac{\bar{\omega}^2}{1+\zeta^2}=c^2. \qquad \dots \quad (6)$$

As the thin layer of liquid is encased between two confocal spheroids, the depth h is given by the relation.

$$h=\frac{d\zeta}{h_2}=\frac{\kappa}{h_2}; \qquad \dots \quad (7)$$

where κ is a constant. If χ be the height of the liquid above the undisturbed surface, the pressure at a height H above the surface of the spheroid is given by

$$\frac{\mathrm{P}}{\rho}=\mathrm{C}+g(h+\chi-\mathrm{H}), \qquad \dots \quad (8)$$

with the usual assumptions that the pressure is the same as the hydrostatic pressure and the variation of gravity along h is neglected.

Now $$g=+\left[\frac{\partial \mathrm{V}}{\partial n}\right]_{\zeta=a}=+\left[h_2\frac{\partial \mathrm{V}}{\partial \zeta}\right]_{\zeta=a}. \qquad \dots \quad (9)$$

As V is a function of ζ only, $\frac{\partial \mathrm{V}}{\partial n}$ is a constant on the surface of the spheroid. Putting

$$f=+\left[\frac{\partial \mathrm{V}}{\partial \zeta}\right]_{\zeta=a},$$

we have

$$g=+h_2f, \text{ where } f \text{ is a constant.} \qquad \dots \quad (10)$$

We have, therefore,

$$\frac{\mathrm{P}}{\rho}=\mathrm{Const.}+fh_2\left(\frac{\kappa}{h_2}+\chi\right),$$

$$=\mathrm{Const.}+fh_2\chi. \qquad \dots \quad (11)$$

§ 4. Denoting by u and w the velocities in the direction of ξ and ϕ respectively, the hydrodynamical equations are

$$\frac{\partial u}{\partial t}=-\frac{h_1}{\rho}\cdot\frac{\partial P}{\partial \xi}-h_1\frac{\partial \Omega}{\partial \xi}=-fh_1\frac{\partial}{\partial \xi}\left(h_2\chi+\frac{\Omega}{f}\right), \quad \ldots (12)$$

$$\frac{\partial w}{\partial t}=-\frac{h_3}{\rho}\cdot\frac{\partial P}{\partial \phi}-h_3\frac{\partial \Omega}{\partial \phi}=-fh_3\frac{\partial}{\partial \phi}\left(h_2\chi+\frac{\Omega}{f}\right), \quad \ldots (13)$$

where Ω is the potential of the disturbing forces.

The equation of continuity is

$$-\frac{\partial}{\partial \xi}\left(h\frac{\delta\phi}{h_3}\,u\right)\delta\xi-\frac{\partial}{\partial \phi}\left(h\frac{\delta\xi}{h_1}\,w\right)\delta\phi$$

$$=\frac{d}{dt}\left(\frac{\delta\xi}{h_1}\,\frac{\delta\phi}{h_3}\,\chi\right),$$

or

$$\frac{1}{h_3h_1}\,\frac{d\chi}{dt}=-\kappa\frac{\partial}{\partial \xi}\left(\frac{u}{h_2h_3}\right)-\kappa\frac{\partial}{\partial \phi}\left(\frac{w}{h_1h_2}\right). \quad \ldots (14)$$

Differentiating (14) with respect to t and substituting from (12) and (13), we have the equation satisfied by $h_2\chi$.

$$\frac{1}{h_1h_2h_3}\,\frac{d^2}{dt^2}(h_2\chi)=f\kappa\left[\frac{\partial}{\partial \xi}\left\{\frac{h_1}{h_2h_3}\,\frac{\partial}{\partial \xi}\left(h_2\chi+\frac{\Omega}{f}\right)\right\}\right.$$

$$\left.+\frac{\partial}{\partial \phi}\left\{\frac{h_3}{h_1h_2}\,\frac{\partial}{\partial \phi}\left(h_2\chi+\frac{\Omega}{f}\right)\right\}\right]; \quad \ldots (15)$$

or,

$$c^2(\xi^2+\zeta^2)\frac{d^2(h_2\chi)}{dt^2}=f\kappa\left[\frac{\partial}{\partial \xi}\left\{(1-\xi^2)\,\frac{\partial}{\partial \xi}\left(h_2\chi+\frac{\Omega}{f}\right)\right\}\right.$$

$$\left.+\frac{\partial}{\partial \phi}\left\{\frac{\xi^2+\zeta^2}{(1-\xi^2)(1+\zeta^2)}\cdot\frac{\partial}{\partial \phi}\left(h_2\chi+\frac{\Omega}{f}\right)\right\}\right]. \quad \ldots (16)$$

Assuming the motion is simple harmonic with time-factor $e^{i\sigma t}$, the equation (16) becomes

$$\frac{c^2\sigma^2}{\kappa f}(\xi^2+\alpha^2)h_2\chi+\frac{\partial}{\partial\xi}\left\{(1-\xi^2)\frac{\partial}{\partial\xi}\left(h_2\chi+\frac{\Omega}{f}\right)\right\}$$

$$+\frac{\xi^2+\alpha^2}{(1-\xi^2)(1-\alpha^2)}\frac{\partial^2}{\partial\phi^2}\left(h_2\chi+\frac{\Omega}{f}\right)=0, \qquad \dots \quad (17)$$

ζ being a constant equal to α.

§ 5. Considering the case of free oscillations, we have putting $\Omega=0$,

$$\frac{c^2\sigma^2}{\kappa f}(\xi^2+\alpha^2)(h_2\chi)+\frac{\partial}{\partial\xi}\left\{(1-\xi^2)\frac{\partial(h_2\chi)}{\partial\xi}\right\}$$

$$+\frac{\xi^2+\alpha^2}{(1-\xi^2)(1+\alpha^2)}\frac{\partial^2(h_2\chi)}{\partial\phi^2}=0. \qquad \dots \quad (18)$$

We may further take $h_2\chi\propto\frac{\cos}{\sin}n\phi$, the equation reducing to the form

$$\frac{d}{d\xi}\left\{(1-\xi^2)\frac{d(h_2\chi)}{\partial\xi}\right\}+(\xi^2+\alpha^2)\left\{\frac{c^2\sigma^2}{\kappa f}\right.$$

$$\left.-\frac{n^2}{(1-\xi^2)(1+\alpha^2)}\right\}h_2\chi=0. \qquad \dots \quad (19)$$

This is a linear differential equation with constant coefficients. The singularities are $\xi=\pm1$. These points are however regular.

For a globe of radius a,

$$\alpha=\infty,\ c=0,\ \text{while } c\alpha=a,$$

this equation reduces to equation (1) of Art 199 of Lamb's Hydrodynamics.

§ 6. Taking the particular case of oscillations symmetrical about the axis we have putting $n=0$,

$$\frac{d}{d\xi}\left\{(1-\xi^2)\frac{d(h_2\chi)}{d\xi}\right\}+(\xi^2+\alpha^2)\frac{c^2\sigma^2}{\kappa f}(h_2\chi)=0. \qquad \dots \quad (20)$$

Putting $A = \frac{c^2\sigma^2}{\kappa f}$ and assuming as a solution

$$h_2\chi = \xi^r(a_0 + a_1\xi + a_2\xi^2 + \ldots),$$

we have as the Indicial equation

$$r(r-1) = 0. \qquad \ldots \quad (21)$$

The successive coefficients are given by the relation

$$a_n = \frac{\{(n-1)(n-2) + Aa^2\}a_{n-2} + Aa_{n-4}}{n(n-1)}\ , \text{ when } n \geq 4 \cdots \quad (22)$$

$$a_n = \frac{\{(n-1)(n-2+Aa^2\}a_{n-2}}{n(n-1)}\ , \text{ when } n < 4. \qquad \ldots \quad (23)$$

We may therefore write the solution in the form

$$h_2\chi = \{a_0 + a_2\xi^2 + a_4\xi^4 + \ldots\} + \{a_1\xi + a_3\xi^3 + a_5\xi^5 + \ldots\} \qquad \cdots \quad (24)$$

§ 7. Consider the disturbing body moving in the plane of the equator at a distance R from the centre. If γ be the gravitational constant, M the mass, R the radius vector, ϕ' the angle between the radius vector and the y-axis, the potential Ω at a point on the surface is given by the equation

$$\Omega = -\frac{\gamma M}{\{x^2 + (R\cos\phi' - y)^2 + (R\sin\phi' - z)^2\}^{1/2}}$$

$$= -\frac{\gamma M}{R}\left\{1 + \frac{y\cos\phi' + z\sin\phi'}{R} - \frac{1}{2}\frac{x^2 + y^2 + z^2}{R^2}\right.$$

$$\left. + \frac{3}{2}\frac{(y\cos\phi' + z\sin\phi')^2}{R^2}\right\} \qquad \ldots \quad (25)$$

neglecting terms of higher order in $\left(\frac{1}{R}\right)$.

The second term represents the potential of a uniform force $\frac{\gamma M}{R^2}$ in the direction of the disturbing body. The potential of the relative attraction is, therefore,

$$\frac{\gamma M}{R^3}\left\{\frac{1}{2}(x^2 + y^2 + z^2) - \frac{3}{2}(y\cos\phi' + z\sin\phi')^2\right\}.$$

Substituting this value in equation (17), we get the height of the tidal wave, though the equation becomes unmanageable.

§ 8. If the spheriod has an angular velocity ω about the x-axis which is taken into account, the problem becomes more complicated. The surface of the spheroid is, as before, an equipotential surface under the gravitational force and the centrifugal force. In fact, if the spheroid is taken to be homogeneous the surface is that of a solidified MacLaurin's spheroid. Making this assumption and taking the equation (1) as the equation of the surface, the gravitational potential at the surface can be written in the form

$$\Omega=\pi\rho\{\alpha_0 x^2+\beta_0 y^2+\beta_0 z^2-\chi_0\} \qquad \dots \quad (26)$$

where $$\alpha_0=abc\int_0^\infty \frac{du}{(a^2+u)\Delta}, \quad \beta_0=abc\int_0^\infty \frac{du}{(b^2+u)\Delta},$$

$$\chi_0=abc\int_0^\infty \frac{du}{\Delta}, \text{ and } \Delta^2=(a^2+u)(b^2+u)^2. \qquad \dots \quad (27)$$

The condition that the surface of the spheroid is an equipotential gives the familiar condition

$$a^2\alpha_0=b^2\left(\beta_0-\frac{\omega^2}{2\pi\rho}\right). \qquad \dots \quad (28)$$

Neglecting the variation of gravity for a small depth h, we may take the potential at the free surface

$$V=V_0+gh, \qquad \dots \quad (29)$$

where $$g=\frac{\partial V}{\partial n}.$$

If l, m, n be the direction-cosines of the normal at x, y, z

$$l=\frac{px}{a^2}, \qquad m=\frac{py}{b^2}, \qquad n=\frac{pz}{c^2},$$

p being the perpendicular from the centre on the tangent surface.

Since the free surface must be an equipotential, we must have

$$gh=c \qquad \dots \quad (30)$$

But

$$lg=2\pi\rho a_0 x \qquad =\lambda\frac{x}{a^2},$$

$$mg=2\pi\rho\left(\beta_0-\frac{\omega^2}{2\pi\rho}\right)y=\lambda\frac{y}{b^2},$$

$$ng=2\pi\rho\left(\beta_0-\frac{\omega^2}{2\pi\rho}\right)z=\lambda\frac{z}{b^2},$$

by virtue of equation (28), λ being a constant.

We have, therefore,

$$= ,$$

and

$$h=kp, \qquad \dots \quad (31)$$

where k is a constant.

§ 9. The equation of the meridian section being

$$\frac{x^2}{c^2\zeta^2}+\frac{\varpi^2}{c^2(1+\zeta^2)}=1$$

$$\frac{1}{p^2}=\frac{x^2}{c^4\zeta^4}+\frac{\varpi^2}{c^4(1+\zeta^2)^2}=\frac{\xi^2+\zeta^2}{c^2\zeta^2(1+\zeta^2)}$$

and

$$\cos\theta=\frac{px}{c^2\zeta^2}=\frac{\xi(1+\zeta^2)^{1/2}}{(\xi^2+\zeta^2)^{1/2}}.$$

The dynamical equations become (writing u for velocity in the direction of ξ and v for velocity in the direction of ϕ)

$$\frac{\partial u}{\partial t}-2\omega v\cos\theta=-h_1\frac{\partial}{\partial\xi}(Z-\bar{Z})g$$

$$\frac{\partial v}{\partial t}+2\omega u\cos\theta=-h_2\frac{\partial}{\partial\phi}(Z-\bar{Z})g \qquad \dots \quad (32)$$

$\bar{Z}$ being the equilibrium-height of the liquid.

The equation of continuity is

$$\frac{1}{h_1 h_3}\frac{dZ}{dt} = -\frac{\partial}{\partial \xi}\left(\frac{kup}{h_3}\right) - \frac{\partial}{\partial \phi}\left(\frac{kvp}{h_1}\right). \quad \ldots \quad (33)$$

For free oscillations, symmetrical about the axis,

$$\bar{Z}=0,\ \frac{\partial Z}{\partial \phi}=0;$$

we have

$$\frac{\partial u}{\partial t} - 2\omega v \cos\theta = -h_1 \frac{\partial (Zg)}{\partial \xi},$$

$$\frac{\partial v}{\partial t} + 2\omega u \cos\theta = 0. \quad \ldots \quad (34)$$

Eliminating v,

$$\frac{\partial^2 u}{\partial t^2} + 4\omega^2 u \cos^2\theta = -h_1 \frac{\partial^2 (Zg)}{\partial \xi \partial t}. \quad \ldots \quad (35)$$

We have therefore as the Particular Integral, the exponential time factor $e^{i\sigma t}$ being understood

$$u = \frac{h_1 \dfrac{\partial^2 (Zg)}{\partial \xi \partial t}}{\sigma^2 - 4\omega^2 \cos^2\theta}, \quad \ldots \quad (36)$$

The complementary function is

$$u = A \cos(2\omega \cos\theta\, t + \epsilon). \quad \ldots \quad (37)$$

Substituting in equation (33)

$$\frac{1}{h_3 h_1}\frac{\partial Z}{\partial t} = -k\frac{\partial}{\partial \xi}\left(\frac{h_1 p}{h_3}\frac{\dfrac{\partial^2 (Zg)}{\partial \xi \partial t}}{\sigma^2 - 4\omega^2\cos^2\theta}\right).$$

Putting η for $\dfrac{\partial Z}{\partial t}$,

$$\frac{\eta}{h_3 h_1} = -k\frac{d}{d\xi}\left(\frac{h_1}{h_3}\cdot\frac{p}{\sigma^2 - 4\omega^2\cos^2\theta}\cdot\frac{d(g\eta)}{d\xi}\right). \quad \ldots \quad (38)$$

Substituting for h_1 and h_3, we get

$$c^2(\xi^2+\zeta^2)^{\frac{1}{2}}(1+\zeta^2)^{\frac{1}{2}}\eta=-k\frac{d}{d\xi}\left(\frac{c\zeta(1-\xi^2)(1+\zeta^2)}{\xi^2+\zeta^2}\,\frac{1}{\sigma^2-4\omega^2\cos^2\theta}\,\frac{d\eta}{d\xi}\right) \quad (39)$$

ζ being a constant a on the surface of the given spheroid. It is a linear differential equation of the second degree. The solution, however, is too complicated to admit physical interpretation.

§ 10. If we take the presence of the disturbing body into account, we have to substitute the value of $\bar{Z}$, which is the equilibrium-height. It is given by the value of the potential investigated in Art 7 above.

On a theorem of Lie relating to the theory of Intermediate Integrals of partial differential equations of the second order

By

Harendranath Datta

The theorem referred to is the following :—

If a partial differential equation of the second order possesses two independent Intermediate Integrals (of the Monge's type *), *it can be reduced to the form $s=0$ by contact transformation.*

The object of the present paper is to show that the possession of two intermediate integrals is a *sufficient condition but not a necessary condition* for equations of the second order which can be transformed into the form $s=0$ by contact transformation. For this purpose, it is enough to find at least one example in which the equation of the second order satisfies the following conditions :—

I. It is reducible to the form $s=0$.

II. The transformation used is a contact transformation.

III. It does not possess two independent intermediate integrals of the Monge's type.

The equation found to satisfy the above conditions is the well-known equation of the Minimal surfaces, *viz.*,

$$(1+q^2)r-2pqs+(1+p^2)t=0.$$

I

Taking Weierstrass's solution of the equation

$$(1+q^2)r-2pqs+(1+p^2)t=0, \qquad \dots \quad \text{(i)}$$

we have

$$x=(1-u^2)U''+2uU'-2U+(1-v^2)V''+2vV'-2V \qquad \dots \quad \text{(ii)}$$

$$y=i[-(1+u^2)U''+2uU'-2U+(1+v^2)V''-2vV'+2V] \qquad \text{(iii)}$$

$$z=2uU''-2U'+2vV''-2V' \qquad \dots \quad \text{(iv)}$$

* From Art. 254, page 295 of Forsyth's *Theory of Differential Equations*, Vol. 6, it is clear that Integrals of Monge's type are meant here. The theorem was afterwards discovered independently by Darboux.

where U and V are arbitrary functions of u and v respectively and the dashes denote differentiations with respect to the corresponding parameter.

If, now, we choose co-ordinates X, Y, Z in such a way that $X=u$, $Y=v$ and $Z=z$ [the relations between u, v and x, y being given by the equations (ii) and (iii)], then it is evident that (iv), when transformed, will reduce to $\dfrac{\partial^2 Z}{\partial X \partial Y}=0$ by differentiation.

But (iv) is really the most general solution (expressed in terms of X, Y, Z) of the equation (i).

Hence, the differential equation (i) is *reducible to the form* $s=0$ by the the preceding transformation.

II

We find from (ii), (iii), (iv) and the relations $X=u$, $Y=v$, $Z=z$ that

$$p=\frac{u+v}{1-uv}, \quad q=i\,\frac{u-v}{1-uv},$$

$$X=\frac{-i+i\sqrt{1+p^2+q^2}}{ip-q}, \quad Y=\frac{-i+i\sqrt{1+p^2+q^2}}{q+ip},$$

$$\frac{\partial X}{\partial p}=\frac{(1-uv)(1-u^2)}{2(1+uv)}, \quad \frac{\partial X}{\partial q}=\frac{i(1-uv)(1+u^2)}{2(1+uv)},$$

$$\frac{\partial Y}{\partial p}=\frac{(1-uv)(1-v^2)}{2(1+uv)}, \quad \frac{\partial Y}{\partial q}=\frac{i(1-uv)(1+v^2)}{2(1+uv)},$$

$$\frac{\partial Z}{\partial p}=\frac{1-uv}{1+uv}\{u(1-u^2)U'''+v(1-v^2)V'''\}$$

and $$\frac{\partial Z}{\partial q}=\frac{i(1-uv)}{1+uv}\{u(1+u^2)U'''+v(1+v^2)V'''\}.$$

Hence, $$P\equiv\frac{\partial Z}{\partial X}=2uU'''$$

and $$Q\equiv\frac{\partial Z}{\partial Y}=2vV'''.$$

Now, the transformation used will be a contact transformation if the relation

$$dZ-PdX-QdY=\rho(dz-pdx-qdy) \qquad \ldots \quad \text{(v)}$$

(where ρ does not vanish) is identically satisfied.

In the present case, the relation reduces to

$$PdX+QdY=pdx+qdy$$

if we take $\rho=1$.

It is easy to see that this last relation is identically satisfied as each side of it becomes equal to

$$2uU'''du+2vV'''dv$$

when expressed in terms of u, v, etc.

Hence, the transformation used in reducing the equation (i) to the form $s=0$ is a *contact transformation*, a fact which is at once clear from the consideration that both the sides of (v) are identically zero here.

III

One of the subsidiary systems of equations (Boole's form) for the determination of the Intermediate integrals is the following :—

$$w_1 \equiv \frac{\partial w}{\partial q} - \rho_1 \frac{\partial w}{\partial p} = 0,$$

$$w_2 \equiv \frac{\partial w}{\partial x} + \sigma_1 \frac{\partial w}{\partial y} + (p+\sigma_1 q)\frac{\partial w}{\partial z} = 0,$$

ρ_1 and σ_1 being respectively equal to

$$\frac{-pq+i\sqrt{1+p^2+q^2}}{1+q^2} \text{ and } \frac{-pq-i\sqrt{1+p^2+q^2}}{1+q^2},$$

the roots of the equation

$$(1+q^2)\mu^2+2pq\mu+(1+p^2)=0$$

If w_1 and w_2 co-exist, then

$$w'_3 \equiv w_2[w_1(w)]-w_1[w_2(w)]^*=0$$

by virtue of $w_1=0$ and $w_2=0$. This cannot be unless

$$w_3 \equiv \frac{\partial w}{\partial y} + k\frac{\partial w}{\partial z} = 0, \quad \text{where} \quad k=\frac{pq+i\sqrt{1+p^2+q^2}}{p-iq\sqrt{1+p^2+q^2}}.$$

Hence, we get $w_3=0$ as a new equation. Now, $(w_1, w_2)=0$* by virtue of $w_3=0$.

Also $$w_4 \equiv (w_2, w_3)=0.$$

But $$w'_5 \equiv (w_1, w_3)=\left(\rho \frac{\partial k}{\partial p}-\frac{\partial k}{\partial q}\right)\frac{\partial w}{\partial z}.$$

Hence $$w_5 \equiv \frac{\partial w}{\partial z}=0$$

is a new equation which is necessary to make $w'_5=0$.

We have now four equations, *viz.*, $w_1=w_2=w_3=w_5=0$, in five variables p, q, x, y, z. It is also easy to see that these four equations form a complete Jacobian system.

Hence, there can be only one integral common to $w_1=0$ and $w_2=0$.

Hence, there is no Intermediate Integral (involving an arbitrary function) of the Monge's type. Similarly, it can be seen that the other system does not possess any such Integral.

Hence, we conclude that the possession of two Intermediate Integrals is a *sufficient condition but not a necessary condition* for equations of the second order which can be transformed into the form $s=0$ by contact transformation.

* The notations are the same as those used in Forsyth's *Theory of Differential Equations.*

4

ON AN EXPRESSION FOR $\frac{d}{dn}J_n(x)$

BY

SUBODCHANDRA MITRA

The object of the present note is to find a proof for a known expression for $\frac{d}{dn}J_n(x)$.

It is possible for us to express $\frac{d}{dn}J_n(x)$ in the form

$$\frac{d}{dn}J_n(x)=J_n(x)\log x-\left(\log 2+\frac{\Gamma'(n+1)}{\Gamma(n+1)}\right)J_n(x)$$

$$+\left\{\frac{(n+2)}{(n+1)}J_{n+2}(x)-\frac{(n+4)}{2(n+2)}J_{n+4}(x)+\frac{(n+6)}{3(n+3)}J_{n+6}(x)\right.$$

$$\left.-\ldots+(-)^{r-1}\frac{(n+2r)}{r(n+r)}J_{n+2r}(x)+\ldots\right.$$

$J_n(x)$ satisfies the differential equation

$$\frac{d^2J_n(x)}{dx^2}+\frac{1}{x}\frac{dJ_n(x)}{dx}+\left(1-\frac{n^2}{x^2}\right)J_n(x)=0.$$

Differentiating with respect to n and writing

$$v=\frac{dJ_n(x)}{dn},$$

we have

$$\frac{d^2v}{dx^2}+\frac{1}{x}\frac{dv}{dx}+\left(1-\frac{n^2}{x^2}\right)v=\frac{2n}{x^2}J_n(x). \qquad \ldots \quad (A)$$

To find a solution of (A) we write

$$v=J_n(x)\log x+\{A_nJ_n(x)+A_{n+2}J_{n+2}(x)+A_{n+4}J_{n+4}(x)$$
$$+A_{n+6}J_{n+6}+\ldots+A_{n+2r}J_{n+2r}(x)+\ldots\}$$

Substituting for v in (A) and taking account of the differential equations satisfied by $J_n(x)$, $J_{n+2}(x)$, etc., we have after a little simplification,

$$A_{n+1}\left\{\frac{(n+2)^2-n^2}{x^2}\right\}J_{n+2}(x)+A_{n+4}\left\{\frac{(n+4)^2-n^2}{x^2}\right\}J_{n+4}(x)$$

$$+\dots+A_{n+2r}\left\{\frac{(n+2r)^2-n^2}{x^2}\right\}J_{n+2r}(x)+\dots$$

$$=\frac{2n}{x^2}J_n(x)-\frac{2}{x}J'_n(x). \qquad \dots \quad (B)$$

Making use of the recurrence formula,

$$J'_n(x)=J_{n-1}(x)-\frac{n}{x}J_n(x),$$

we have

$$A_{n+2}\{(n+2)^2-n^2\}J_{n+2}(x)+A_{n+4}\{(n+4)^2-n^2\}J_{n+4}(x)+\dots$$

$$+A_{n+2r}\{(n+2r)^2-n^2\}J_{n+2r}(x)+\dots$$

$$=4nJ_n(x)-2xJ_{n-1}(x).$$

Now

$$xJ_{n-1}(x)=2nJ_n(x)-2(n+2)J_{n+2}(x)+2(n+4)J_{n+4}(x)-\dots$$

Therefore

$$A_{n+2}\{(n+2)^2-n^2\}J_{n+2}(x)+A_{n+4}\{(n+4)^2-n^2\}J_{n+4}(x)+\dots$$

$$+A_{n+2r}\{(n+2r)^2-n^2\}J_{n+2r}(x)+\dots$$

$$=4(n+2)J_{n+2}(x)-4(n+4)J_{n+4}(x)+4(n+6)J_{n+6}(x)-\dots$$

$$+(-)^{r-1}4(n+2r)J_{n+2r}(x)+\dots \qquad \dots \quad (C)$$

Therefore equating the co-efficients, we have,

$$A_{n+2}=\frac{(n+2)}{(n+1)}, \qquad A_{n+4}=+\frac{(n+4)}{2(n+2)},$$

$$\dots \qquad \dots \qquad \dots$$

$$A_{n+2r}=(-)^{r-1}\frac{(n+2r)}{r(n+r)}.$$

It now remains to determine A_n. When n is any number, real or complex

$$J_n(x)=\frac{x^n}{2^n\Gamma(n+1)}\left\{1-\frac{x^2}{2^2 1!(n+1)}+\frac{x^4}{2^4 2!(n+1)(n+2)}-\dots\right\}$$

the expression being rendered precise by taking for x its principal value.

Therefore the co-efficient of x in $\frac{d}{dn}J_n(x)$ is

$$-\left(\frac{\log 2}{2^n\Gamma(n+1)}+\frac{\Gamma'(n+1)}{2^n\{\Gamma(n+1)\}^2}\right)$$

and equating it to the co-efficient of x^n in $A_n J_n(x)$, we have

$$A_n=-\left(\log 2+\frac{\Gamma'(n+1)}{\Gamma(n+1)}\right).$$

In conclusion I wish to express my indebtedness to Dr. N. M. Basu for the interest he always takes in my work and to Dr. A. B. Datta for his kind revision and expression of opinion.

5

A few interesting results in connection with the motion of a heavy inexentensible chain over a fixed vertical pulley

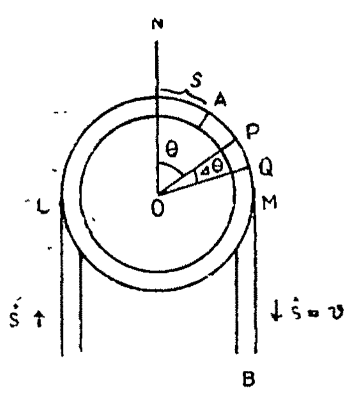
By

A. C. Banerji.

§ 1. Let us first assume that there is no friction between the chain and the pulley, and the resistance of the air is neglected, and the cross section of the chain is small.

It will be an interesting exercise first to find the difference between the tensions at any instant at the two points L, M, where the chain ceases to be in contact with the pulley.

Let a be the radius of the pulley, and let m be the linear density of the chain which is supposed to be constant; let A be a marked point on the chain; let the arc NA be s at the instant t; let P be any other point on the chain and let the length AP be σ; let the element PQ of the chain be $\Delta\sigma$; let $\angle$NOP be θ and $\angle$POQ be $\Delta\theta$.



Let R be the reaction per unit length between the pulley and the chain at the point P at the instant t.

The total reaction on PQ is $R\Delta\sigma$ to the first order of small quantities along a direction making an angle $\Delta\phi$ with OP where $\Delta\phi < \Delta\theta$.

Consider the equation of motion for the element PQ of the chain along the tangent at P.

Each element of the chain has the same velocity and acceleration as the marked point A at any instant

$$m\Delta\sigma\dot{v} = m\Delta\sigma g\sin\theta + (T+\Delta T)\cos\Delta\theta - T + R\Delta\sigma\sin\Delta\phi\,;$$

neglecting small quantities of 2nd order we have,

$$m\Delta\sigma\dot{v} = m\Delta\sigma g\sin\theta + \Delta T.$$

Dividing by $\Delta\sigma$ and proceeding to limit we have,

$$m\dot{v}=mg\sin\theta+\frac{dT}{d\sigma}\cdot \quad \ldots \quad (A)$$

Now again, $a\theta=s+\sigma$,

as s is independent of σ, differentiating along the chain we get

$$ad\theta=d\sigma.$$

Integrate (A) with respect to σ from L to M and let T_1 and T_2 be the tensions at L and M respectively; we have

$$m\dot{v}\int_L^M d\sigma=mg\int_L^M \sin\theta\, d\sigma+\int_L^M dT,$$

$$m\dot{v}\pi a=mga\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta d\theta+T_2-T_1$$

$$\therefore\ m\pi a\dot{v}=T_2-T_1. \quad \ldots \quad (B)$$

Let us examine this equation. There are three interesting cases here.

(*i*) $T_2 \longrightarrow T_1$, when $m \longrightarrow o$;

i.e., the two tensions are the same if the chain is light.

(*ii*) $T_2 \longrightarrow T_1$, when $a \longrightarrow o$;

i.e., the two tensions are the same if the pulley is small even if the chain is not light.

(*iii*) $T_2=T_1$, when $\dot{v}=o$;

i.e. the two tensions are the same if the chain is moving with uniform speed even if the chain is not light and the pulley is not small.

Let us now calculate the total vertical upward pressures on chain.

It is

$$\int_L^M R d\sigma \cos\theta$$

Now the equation of motion of the element PQ of chain along the inward normal at P is

$$m\Delta\sigma\frac{v^2}{a}=m\Delta\sigma g\cos\theta-\mathrm{R}\Delta\sigma\cos\Delta\phi+(\mathrm{T}+\Delta\mathrm{T})\sin\Delta\theta\ ;$$

neglecting small quantities of the 2nd order we have

$$\frac{mv^2}{a}\ \Delta\sigma=m\Delta\sigma g\cos\theta-\mathrm{R}\Delta\sigma+\mathrm{T}\Delta\theta$$

$$=m\Delta\sigma g\cos\theta-\mathrm{R}\Delta\sigma+\frac{\mathrm{T}}{a}\Delta\sigma$$

$$\therefore\ \mathrm{R}=-\frac{mv^2}{a}+mg\cos\theta+\frac{\mathrm{T}}{a},$$

where T is the tension at P

§ 2. Now let us find out T.

Integrate equation (A) with respect to σ from L to P ; we have,

$$m\dot{v}\int_{\mathrm{L}}^{\mathrm{P}}d\sigma=mg\int_{\mathrm{L}}^{\mathrm{P}}\sin\theta d\sigma+\int_{\mathrm{L}}^{\mathrm{P}}d\mathrm{T}$$

$$m\dot{v}\left(\frac{a\pi}{2}+a\theta\right)=mga\int_{-\frac{\pi}{2}}^{\theta}\sin\theta d\theta+\mathrm{T}-\mathrm{T}_1$$

$$=-mga\cos\theta+\mathrm{T}-\mathrm{T}_1$$

$$\therefore\ \mathrm{T}=\mathrm{T}_1+m\dot{v}a\left(\frac{\pi}{2}+\theta\right)+mga\cos\theta\ ;$$

$$\therefore\ \mathrm{R}=-\frac{mv^2}{a}+2mg\cos\theta+\frac{\mathrm{T}_1}{a}+m\dot{v}\left(\frac{\pi}{2}+\theta\right).\qquad\dots\quad(\mathrm{C})$$

Now let the total vertical Thrust on the chain be F. Then

$$F=\int_{L}^{M} \mathrm{R}d\sigma \cos\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{R}a d\theta\cos\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a d\theta \cos\theta \left\{\frac{\mathrm{T}_1-mv^2}{a}+m\dot{v}\left(\frac{\pi}{2}+\theta\right)+2mg\cos\theta\right\}$$

$$=2(\mathrm{T}_1-mv^2)+ma\dot{v}\pi+mga\pi \qquad \cdots \quad (\mathrm{D})$$

$$=\mathrm{T}_1+\mathrm{T}_2-2mv^2+mga\pi \qquad \cdots \quad (\mathrm{E})$$

as $\mathrm{T}_2=\mathrm{T}_1+ma\dot{v}\pi$

Let us also examine this equation ; there are three interesting cases:

(*i*) Let $m\longrightarrow o$, then $\mathrm{F}\longrightarrow 2\mathrm{T}_1$,

from (D), *i.e.*, if the chain be light then the total vertical thrust on the chain is twice the tension at L or M.

(*ii*) Let $v=o$ and $\dot{v}=o$, then $\mathrm{F}=2\mathrm{T}_1+mga\pi$

i.e., if the chain is at rest and continues to be at rest then the total vertical thrust on the chain is twice the tension at L or M together with the weight of the chain in contact with the pulley.

[*N.B.*—If initially $v=o$, but $\dot{v}\neq o$, then

$$\mathrm{F}=\mathrm{T}_1+\mathrm{T}_2+mga\pi$$

from (E), here $\mathrm{T}_2\neq\mathrm{T}_1$

i.e., if the chain initially starts from rest, then the tensions at L and M are different, and the total vertical thrust at that instant on the chain is equal to the sum of the tensions at L and M, together with the weight of the chain in contact with the pulley.]

(*iii*) Let $a\longrightarrow o$ then $\mathrm{F}\longrightarrow 2(\mathrm{T}_1-mv^2)$;

i.e., if the pulley is small, the total vertical thrust on the chain is less than twice the tensions at L or M by $2mv^2$.

§ 3. Let us now consider in the case of a large smooth pulley, if it is possible to make the total vertical thrust between the chain and the pulley vanish.

Let the end B of the chain be free. Let us consider the equation of motion of the portion BM of the chain and let x be measured from M;

$$\therefore\ mxv\frac{dv}{dx}=mgx-T_2.$$

We have also

$$T_2=T_1+ma\dot{v}\pi\ ;$$

$$\therefore\ mxv\frac{dv}{dx}=mgx-T_1-ma\dot{v}\pi$$

$$\therefore\ T_1=-mxv\frac{dv}{dx}+mgx\quad ma\dot{v}\pi.$$

Now the total thrust vanishes when $F=o$

i.e. $$2(T_1-mv^2)+ma\dot{v}\pi+mga\pi=o$$

from (D)

$$\therefore\quad -2xv\frac{dv}{dx}+2gx-2a\dot{v}\pi-2v^2+a\dot{v}\pi+ag\pi=o,$$

Now as $$\dot{v}=v\frac{dv}{dx},$$

we have

$$(2x+a\pi)v\frac{dv}{dx}+2v^2=2g\left(x+\frac{a\pi}{2}\right). \qquad \ldots \quad (G)$$

Put $x+\dfrac{a\pi}{2}=z$; then $zv\dfrac{dv}{dz}+v^2=gz$

$$\therefore\ \frac{d}{dz}(z^2v^2)=2gz^2$$

or

$$z^2v^2=\frac{2gz^3}{3}+\text{const.}$$

If we have the initial conditions that $v=\sqrt{\dfrac{ag\pi}{3}}$, when $z=\dfrac{a\pi}{2}$ *i.e.*, $x=0$, we find the constant to be zero.

§ 4. It is *experimentally* possible to have these initial conditions.

At first we can hold the chain at rest with free end B at M by means of some contrivance applied to the left hand portion of the chain.

Then as soon as the experiment begins, we apply a suitable impulse at B so that a velocity equal to $\sqrt{\frac{ag\pi}{3}}$ is imparted to the chain, and at the same time the contrivance is let go.

If it is objected, that with free end B at M, it is not easy to apply a contrivance to the left hand portion of the chain, before the experiment begins, we can alter these initial conditions a bit and get new initial conditions which will also make the constant zero.

Initially take BM or x to be $\frac{a\pi}{2}$. At first we attach a clip fixed to a stand to the portion BM, and another clip fixed also to a stand is attached to the left hand portion of the chain. As soon as the experiment begins we apply a suitable impulse at B imparting a velocity $\sqrt{\frac{2ga\pi}{3}}$ to the chain and at the same instant two clips are let go.

Under these initial conditions we have the constant zero;

$$\therefore v^2 = \frac{2}{3}gs = \frac{2}{3}g\left(x + \frac{a\pi}{2}\right)$$

$$\therefore v\frac{dv}{dt} = \frac{g}{3}\dot{x} = \frac{g}{3}v$$

$$\therefore \frac{dv}{dt} = \frac{g}{3}$$

This is an interesting and important result. In the case of a large smooth pulley, the vertical thrust between the chain with a free end B and the pulley will vanish if the chain moves with an acceleration equal to one third of gravity, with the initial condition as if a velocity equal to $\sqrt{\frac{ag\pi}{3}}$ is imparted to the chain when the free end B is at M *i.e.* at the same level with the centre of the pulley.

When the radius of the pulley tends to zero, this initial velocity tends to zero. So in the case of a very small pulley we can say without much error that the initial conditions are $v=0$, when $x=0$.

By suitable arrangements on the left hand portion of the chain it is *experimentally* possible to make the chain move with an acceleration equal to one third of gravity.

§ 5. Now let us take the case when a weight equal to that of length "l" of the chain is attached to the free end B of the chain.

The equation of motion for the portion BM of the chain with the weight attached at B, becomes

$$m(x+l)v\frac{dv}{dx}=mg(x+l)-T_2,$$

and equation (G) becomes

$$(2x+2l+a\pi)v\frac{dv}{dx}+2v^2=2g\left(x+l+\frac{a\pi}{2}\right).$$

Put $$z=x+l+\frac{a\pi}{2},$$

then we have

$$v\frac{dv}{dz}+v^2=g\,z$$

$$\therefore\ z^2v^2=\frac{2gz^3}{3}+\text{const}$$

If we have the initial conditions such that

$$v=\sqrt{\frac{g}{3}(a\pi+2l)}$$

when $$z=l+\frac{a\pi}{2},\quad \text{i.e.,}\quad x=o;$$

we have the constant equal to zero.

$$\therefore\ v^2=\frac{2gz}{3}=\frac{2g}{3}\left(x+l+\frac{a\pi}{2}\right).$$

$$\therefore\ \frac{dv}{dt}=\frac{1}{3}g.$$

In the case where a weight equal to that of length l of the chain is attached to the free end B, the vertical thrust between the chain and the pulley will vanish if the chain moves with an acceleration equal to one third of gravity, with the initial condition as if a velocity equal to $\sqrt{\frac{g}{3}\left(a\pi+2l\right)}$ is imparted to the chain when the free end B is at M, *i.e.*, at the same level with the centre of the pulley.

§ 6. Let us now consider the total horizontal thrust on the chain. Let H be the total horizontal thrust. Then

$$H=\int_{L}^{M} Rd\sigma \sin\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Rad\theta \sin\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a\sin\theta \left\{\frac{T_1-mv^2}{a}+m\dot{v}\left(\frac{\pi}{2}+\theta\right)+2mg\cos\theta\right\}d\theta$$

$$=2ma\dot{v}$$

Now let us examine the three cases :—

(*i*) When $m \to o$, $H \to o$;

i.e., when the chain is light, the total horizontal thrust vanishes even if the pulley be not small.

(*ii*) When $a \to o$, $H \to o$;

i.e. when the pulley is small, the total horizontal thrust vanishes even if the chain be not light

(*iii*) When $\dot{v}=o$, $H \to o$;

i.e., when the chain moves with uniform speed the total horizontal thrust vanishes.

[*N.B.* By means of suitable arrangements it is not impossible to make the chain move with uniform speed.]

6

New methods of approximating to the roots of a numerical equation

By

Nripendranath Ghosh

The object of this paper is to develop a new method and to indicate others by which the solution of any numerical equation can be approximated to. The procedure adopted embraces some methods already known and of these methods special interest has been attached in this paper to those of Newton, Horner and McClintock. It will appear in course of development that certain improvements have been made or attempted with regard to each of the above-mentioned methods.

The underlying principle

1. We shall consider for simplicity of treatment rational integral numerical equations. Let $\phi(z)=p$ be such an equation of degree n. Let there be a root differing by a small quantity from a number a. We shall describe methods by which the required root may be calculated to any desired degree of approximation.

Denoting by x the number $p-\phi(a)$, the above equation can be put in the form $\phi(z)=\phi(a)+x$, where we call x the *residue* of the equation corresponding to a. The ultimate process of obtaining the root depends now upon the following fundamental theorem :—

If $a_0, a_1, a_2, \ldots a_s$ and ξ be arbitrary parameters there exists a rational integral identity

$$\phi(a_0+a_1\xi+a_2\xi^2+\ldots+a_s\xi^s)=A_0+A_1\xi+A_2\xi^2+\ldots+A_{sn}\xi^{sn} \quad \ldots \quad (A)$$

where A_0, A_1, $A_2 \ldots$ are known rational integral functions of a_0, a_1, $a_2 \ldots$ given by

$$A_0=\phi(a_0)$$

$$A_1=a_1\phi'(a_0)$$

$$\underline{|2}A_2=a_1{}^2\phi''(a_0)+2a_2\phi'(a_0)$$

$$\underline{|3}A_3=a_1{}^3\phi'''(a_0)+6a_1a_2\phi''(a_0)+6a_3\phi'(a_0)$$

$$\underline{|4}A_4 = a_1^4\phi^{iv}(a_0)+12a_1^2a_2\phi'''(a_0)+\{12a_2^2+24a_1a_3\}\phi''(a_0)$$
$$+24a_4\phi'(a_0)$$

... ...

$$\underline{|s}A_s = a_1^s\,\phi^{(s)}(a_0)+s(s-1)a_1^{s-2}a_2\,\phi^{(s-1)}(a_0)+\dots+\underline{|s}a_s\phi'(a_0)$$

... ...

$$\underline{|n-1}A_{s,n-1} = a_s^{n-1}a_{s-1}\phi^{(n)}(a_0)$$

$$\underline{|n}A_{s,n} = a_s^n\,\phi^{(n)}(a_0).$$

A number of interesting relations existing among the co-efficients $A_0, A_1, A_2 \dots$ has been given in my paper entitled "Algebra of Polynomials" Chapter II.* Of these we give here two only which facilitate the successive calculation of the A's.

(1) $$(r+1)A_{r+1} = \Delta_{s0}A_r,$$

where Δ_{s0} stands for the linear differential operator

$$a_1\frac{\partial}{\partial a_0}+2a_2\frac{\partial}{\partial a_1}+3a_3\frac{\partial}{\partial a_2}+\dots+sa_s\frac{\partial}{\partial a_{s-1}}$$

(2) $$(r+1)A_{r+1} = \frac{\partial}{\partial a_0}\{a_1A_r+2a_2A_{r-1}+\dots$$
$$+ra_rA_1+(r+1)a_{r+1}A_0\}$$

2. To obtain the required root of the equation $\phi(z)=\phi(a)+x$ by means of the foregoing theorem, we have to adopt a process of inversion, *i.e.*, we have to find values for each of $a_0, a_1\xi, a_2\xi^2, \dots a_s\xi^s$ in such a way that the expression $A_0+A_1\xi+A_2\xi^2+\dots+A_{sn}\xi^{sn}$ in (A) may differ from $\phi(a)+x$ by a quantity which can be made to vanish by sufficiently increasing s. It is clear that the mode in which the above expression (henceforth to be denoted simply by u_A) may be made to approach $\phi(a)+x$ is not unique and accordingly different methods of solving numerical equations can be attempted.

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Horner's Method

3. The method due to Horner of solving numerical equations is one of these where the above principle is maintained. Before passing on to indicate other methods we wish to consider this straight-forward one with special reference to the particular mode in which the expression u_A is made to approach $\phi(a)+x$. Avoiding details of the process, as being unnecessary here, we may proceed thus :

The first trial a, having been made with regard to the required root of the equation $\phi(z)=p$, we have

$$\phi(a)+x=p \qquad \ldots \quad (i)$$

Put then $a_0=a$ in the identity (A) and consider the *collection* of terms of the expression u_A represented by $\phi(a+a_1\xi)-\phi(a)$. Form the equation $\phi(a+a_1\xi)-\phi(a)=x$ and let a_1x be an approximate value of $a_1\xi$ (found by suitable trials) of the above equation, so that

$$\phi(a+a_1x)-\phi(a)=x-x_1, \quad \text{where } |x_1|<|x| \qquad \ldots \quad (ii)$$

Put then $a_1\xi=a_1x$ in the identity (A) and consider the terms of u_A contained in the next *collection* $\phi(a+a_1x+a_2\xi^2)-\phi(a+a_1x)$. Form the equation $\phi(a+a_1x+a_2\xi^2)-\phi(a+a_1x)=x_1$ and let a_2x^2 be an approximate value of $a_2\xi^2$ (found as before by trials) in this equation, so that

$$\phi(a+a_1x+a_2x^2)-\phi(a+a_1x)=x_1-x_2, \text{ where } |x_2|<|x_1| \quad (iii)$$

Put then $a_2\xi^2=a_2x^2$ in the identity (A) and proceed as before. The $(s+1)$th relation thus obtained will be

$$\phi(a+a_1x+a_2x^2+\ldots+a_sx^s)-\phi(a+a_1x+a_2x^2+\ldots+a_{s-1}x^{s-1})$$

$$=x_{s-1}-x_s, \text{ where } |x_s|<|x_{s-1}| \ldots \quad (s+i)$$

Combining the relations (i), (ii), $(iii)\ldots(s+i)$ we have

$$\phi(a+a_1x+a_2x^2+\ldots a_sx^s)=p-x_s$$

where x_s represents the *residue* corresponding to

$$a+a_1x+a_2x^2+\ldots+a_sx^s$$

Since $|x|$, $|x_1|$, $|x_2|$ ... $|x_r|$ form a sequence of decreasing numbers, the approach of u_A towards $\phi(a)+x$, *i.e.*, p is definite and certain.

4. The successive *collections* of terms in u_A associated with the above process mark the path of approach of u_A towards $\phi(a)+x$. The manner of collecting resorted to in Horner's method is exhaustive; for in considering a certain collection say...

$$\phi(a+a_1x+a_2x^2+a_3\xi^3)-\phi(a+a_1x+a_2x^2)$$

of u_A we notice that all the terms involving the unknown $a_3\xi^3$ are included. Such collections from u_A will be called *complete* while others containing lesser number of terms involving the unknown will be called *partial*. A partial collection having a single term involving the unknown is said to be *simple*, otherwise it is called *multiple*. The collections represented by $A_1\xi$, $A_2\xi^2$, $A_3\xi^3$... involving respectively the unknowns $a_1\xi$, $a_2\xi^2$, $a_3\xi^3$... are all *simple partial collections*. Since they present themselves naturally in the formation of the identity (A) we shall call these the *natural system of partial collections*.

A New Method by Series

5. Proceeding in succession along this natural system of collections we develop a new method of solving numerical equations by means of series in the following way :—

Put $a_0=a$ in the identity (A) and consider the collection $A_1\xi$. Form the equation $A_1\xi=x$ linear in $a_1\xi$; whence $a_1\xi=\frac{x}{\phi'(a)}$ (provided $\phi'(a)\neq 0$). Put then $a_1\xi=\frac{x}{\phi'(a)}$ in the identity (A) and consider the next collection $A_2\xi^2$. Form the equation $A_2\xi^2=0$ linear in $a_2\xi^2$ whence

$$a_2\xi^2=-\frac{x^2}{\underline{|2}}\cdot\frac{\phi''(a)}{\{\phi'(a)\}^3}:$$

Substitute this value of $a_2\xi^2$ in the identity (A) and form the next equation $A_3\xi^3=0$; whence

$$a_3\xi^3=\frac{x^3}{\underline{|3}}\cdot\frac{3\{\phi''(a)\}^2-\phi'(a)\phi'''(a)}{\{\phi'(a)\}^5}$$

Continuing in this manner we come upon the collection represented by $A_s\xi^s$, whence we choose $a_s\xi^s$ such that $A_s\xi^s=0$, the group of terms

$$-(A_{s+1}\xi^{s+1}+A_{s+2}\xi^{s+2}+\ldots+A_{sn}\xi^{sn})$$

representing the corresponding residue. So long as the residues tend to become numerically smaller and smaller as s increases, the required root is given more and more accurately by the $(s+1)$ terms of the infinite series

$$a+\frac{x}{\phi'(a)}-\frac{x^2}{\underline{|2}}\cdot\frac{\phi''(a)}{\{\phi'(a)\}^3}+\frac{x^3}{\underline{|3}}\cdot\frac{3\{\phi''(a)\}^2-\phi'(a)\phi'''(a)}{\{\phi'(a)\}^5}-\ldots$$

If C_r and C_{r+1} be the co-efficients of $\frac{x^r}{\underline{|r}}$ and $\frac{x^{r+1}}{\underline{|r+1}}$ respectively in this series then it will be found that

$$\left(\frac{1}{\phi'(a)}\,\frac{d}{da}\right)C_r=C_{r+1}$$

We shall represent the above series by the symbol ${}^{x}\rho_{\phi}$ or simply by ρ and the value calculated up to its $(s+1)$th term by a_s. Let R_s denote the residue corresponding to a_s; then we have the obvious relation $\phi(a_s)+R_s=\phi(a)+x$.

Newton's Method

6. Sometimes it is convenient to have the equation in the form $\phi(z)=0$. To find the required root we put $x=-\phi(a)$ in the series ${}^{x}\rho_{\phi}$ and obtain the following series

$$a-\frac{\phi(a)}{\phi'(a)}-\frac{\{\phi(a)\}^2}{\underline{|2}}\cdot\frac{\phi''(a)}{\{\phi'(a)\}^3}$$

$$-\frac{\{\phi(a)\}^3}{\underline{|3}}\cdot\frac{3\{\phi''(a)\}^2-\phi'(a)\phi'''(a)}{\{\phi'(a)\}^5}-\ldots$$

represented by ${}^{-\phi}\rho_{\phi}$.

If we take only the first two terms of the series ${}^{-\phi}\rho_{\phi}$ then by the process of iteration there follows the method of solving an equation of the form $\phi(z)=0$ due to Newton. Since now we have found the complete series ${}^{-\phi}\rho_{\phi}$ we may apply the process of iteration beyond the second term. If, however, the series ${}^{-\phi}\rho_{\phi}$ be not slowly convergent the process of iteration may with advantage be altogether dispensed with.

7. Let us illustrate the application of the series ${}^{x}\rho_{\phi}$ or ${}^{-\phi}\rho_{\phi}$ by considering the following examples :—

Ex. (*i*) Solve $z^3-2z-5=0.$

Suppose we are going to find the root lying between 2 and 3.

Put the equation in the form $\phi(z)=p$, so that $\phi(z)=z^3-2z$ and $p=5$ choose $a=2$, then $\phi(a)=4$ and $p-\phi(a)=x=1$.

Also
$$\phi'(a)=10,$$
$$\phi''(a)=12,$$
$$\phi'''(a)=6,$$
$$\phi^{\prime\prime\prime\prime}(a)=0$$

Substituting these values in the series ${}^{x}\rho_{\phi}$ we have

$$a_3=2+\frac{1}{10}-\frac{1}{\underline{|2}}\cdot\frac{12}{10^3}+\frac{1}{\underline{|3}}\cdot\frac{3.12^2-10.6}{10^5}$$

$$=2+\cdot 1-\cdot 006+\cdot 00062$$

$$=2\cdot 09462,$$

which gives the root correct to the third decimal figure. Proceeding to $a_4, a_5, a_6 \ldots$ we get more and more accurate values. For a closer choice $2\cdot 1$ of a, even a_3 gives the root correct to the 7th or 8th figure.

Ex. (*ii*) Solve $z^4+4z^3-4z^2-11z+4=0$.

Let us find the root lying between 1 and 2.

Take $$\phi(z)=z^4+4z^3-4z^2-11z+4,$$

then $$\phi'(z)=4z^3+12z^2-8z-11,$$

$$\phi''(z)=12z^2+24z-8,$$

$$\phi'''(z)=24z+24,$$

$$\phi^{iv}(z)=24,$$

and the equation has the form $\phi(z)=0$.

(*a*) Choose $a=1$, then

$$\phi(a)=-6, \qquad \phi'(a)=-3, \qquad \phi''(a)=28,$$

$$\phi'''(a)=48, \qquad \phi^{iv}(a)=24.$$

The ratio $\dfrac{\phi(a)}{\phi'(a)}$ being greater than 1, the series $^{-\phi}\rho_{\phi}$ obviously cannot afford a root.

(*b*) Choose $a=2$, then

$$\phi(a)=14; \qquad \phi'(a)=53, \qquad \phi''(a)=88,$$

$$\phi'''(a)=72 \qquad \phi^{iv}(a)=24.$$

Substituting these values in $^{-\phi}\rho_{\phi}$ we have

$$a_3=2-\frac{14}{53}-\frac{14^2}{\underline{|2}}\cdot\frac{88}{53^3}-\frac{14^3}{\underline{|3}}\cdot\frac{3.88^2-53.72}{(53)^5}\ {}^*$$

$$=2-\cdot264151-\cdot057927-\cdot021233$$

$$=1\cdot65\ldots,$$

where we can scarcely depend upon its first decimal figure, the series being slowly convergent. We, however, see that $1\cdot6$ is a closer choice for a than 2.

* In numerical calculations logarithmic tables have been used.

(c) When $a=1\cdot 6$

$$\phi(a)=-\cdot 9024, \qquad \phi'(a)=23\cdot 304, \qquad \phi''(a)=61\cdot 12,$$

$$\phi'''(a)=62\cdot 4, \qquad \phi^{iv}(a)=24.$$

Now a_3 becomes

$$1\cdot 6-\frac{-\cdot 9024}{23\cdot 304}-\frac{(\cdot 9024)^2}{\lfloor 2}\cdot\frac{61\cdot 12}{(23\cdot 304)^3}$$

$$+\frac{(\cdot 9024)^3}{\lfloor 3}\cdot\frac{3(61\cdot 12)^2-23\cdot 304\times 62\cdot 4}{(23\cdot 304)^5},$$

or $$1\cdot 6+\cdot 038723-\cdot 0019663+\cdot 0001738,$$

or $$1\cdot 63693,$$

which gives the root correct even up to the fourth decimal figure.

In order that the series ${}^{x}\rho_{\phi}$ or ${}^{-\phi}\rho_{\phi}$ (as the case may be) may afford a root it is essential that the series must not be slowly convergent. The values of a for which this is maintained are in general ranged within limits either wide or narrow. For want of a *simple convergency-criterion* of the above series these limits cannot be definitely pre-assigned.

8. Referring to Art. 5 we observe that the series ${}^{x}\rho_{\phi}$ was formed under the tacit assumption that $A_1\xi$ is numerically the greatest *collection* in u_A. If $|a_1\xi|$ is *sufficiently* small it is in general so irrespective of the values of $\phi'(a)$, $\phi''(a)$. $\phi'''(a)\ldots$* Since $a_1\xi=\frac{x}{\phi'(a)}$ it follows that $|x|$ should be sufficiently small. Thus the series ${}^{x}\rho_{\phi}$ must lead to a root provided a can be so chosen that the residue corresponding to it is *sufficiently* small in numerical value.

If $|a_1\xi|$ is only less than 1, but not *sufficiently* small, unless $\phi'(a)$ happens to be large enough we cannot expect $A_1\xi$ to be the greatest collection. The term $\frac{(a_1\xi)^2}{\lfloor 2}\phi''(a)$, for instance may exceed $A_1\xi$ in

* $|a_1\xi|$, $|a_2\xi^2|$, $|a_3\xi^3|$. necessarily forming a sequence of decreasing numbers.

numerical value if $|\phi'(a)|$ tend to become small while $|\phi''(a)|$ tend to increase. The series ${}^{r}\rho_{\phi}$ will fail in a case like this. A different mode of collecting the terms of u_A yields, however, an auxiliary series applicable in such a case. We shall describe the method in the next article.

Returning for a moment to Horner's method it may be remarked that the series ${}^{-\phi}\rho_{\phi}$ can be applied at some stage in the method to reduce the labour entailed in the process. The chief labour consists in the calculation of x's at successive stages of the method and it must be remembered that these calculations are intimately related with the process. In practice, the application of the series ${}^{-\phi}\rho_{\phi}$ is most convenient when the *trial divisor* is effective (that is the residue is *sufficiently* small).

Case of failure of the series ρ

9. Let us first explain the particular mode of collecting terms required in this process. Take the term $\frac{(a_1\xi)^2}{\lfloor 2}\phi''(a_0)$ in $A_2\xi^2$ and form the first collection K_1. Next take the term $A_1\xi$ along with all but the last in $A_3\xi^3$ and form the second collection K_2. Thus K_2 will consist of the terms

$$a_1\xi\phi'(a_0)+\frac{(a_1\xi)^3}{\lfloor 3}\phi'''(a_0)+a_1\xi a_2\xi^2\phi''(a_0).$$

Next take the last term in $A_2\xi^2$ along with all but the last in $A_4\xi^4$ and form the third collection K_3. Finally the sth collection K_s will involve the last term in $A_{s-1}\xi^{s-1}$ together with all the terms in $A_{s+1}\xi^{s+1}$. The successive partial collections thus formed are all simple.

Put then $a_0=a$ in the identity (A) and choose $a_1\xi$, $a_2\xi^2$, $a_3\xi^3 \dots a_s\xi^s$ such that $K_1=x$, $K_2=0$, $K_3=0, \dots K_s=0$; we then have

$$\phi(a_0+a_1\xi+a_2\xi^2+\dots+a_s\xi^s)$$

$$=\phi(a)+x+a_s\xi^s\phi'(a)+A_{s+2}\xi^{s+2}+A_{s+3}\xi^{s+3}+\dots+A_{sn}\xi^{sn}$$

where $a_0 = a$,

$$a_1\xi = \pm\left\{\frac{2x}{\phi''(a)}\right\}^{\frac{1}{2}},$$

$$a_2\xi^2 = -\frac{\phi'(a)}{\phi''(a)} - \frac{1}{3}\cdot\frac{x\phi'''(a)}{\{\phi''(a)\}^2},$$

$$a_3\xi^3 = \pm\left\{\frac{\phi''(a)}{2x}\right\}^{\frac{1}{2}}\left[\frac{1}{2}\,\frac{\{\phi'(a)\}^2}{\{\phi''(a)\}^2} + \frac{x\phi'(a)\phi'''(a)}{\{\phi''(a)\}^3}\right.$$

$$\left. - \frac{1}{6}\frac{x^2\phi^{iv}(a)}{\{\phi''(a)\}^3} + \frac{5}{18}\,\frac{x^2\{\phi'''(a)\}^2}{\{\phi''(a)\}^4}\right];$$

and so on.

The process of inversion may thus be performed in two ways under the tacit assumption that K_1 is numerically the greatest collection among the K's. So long as the residue

$$-\{a_s\xi^s\phi'(a) + A_{s+2}\xi^{s+2} + \ldots + A_{ss}\xi^{ss}\}$$

tends in each case to become smaller and smaller in numerical value as s increases, a pair of roots of the equation $\phi(s) = \phi(a) + x$ is given more and more accurately by the $(s+1)$ terms of the infinite series

$$a \pm \left\{\frac{2x}{\phi''(a)}\right\}^{\frac{1}{2}} - \left\{\frac{\phi'(a)}{\phi''(a)} + \frac{1}{3}\,\frac{x\phi'''(a)}{\{\phi''(a)\}^2}\right\} \pm \cdots$$

We shall represent the above by $\overset{\pm x^{\frac{1}{2}}}{\Sigma}_{\phi}$ or simply by Σ. In order that the pair may be real we must have x and $\phi''(a)$ of the same sign.

10. As an illustration of the application of the series Σ let us consider the case (a) of *Ex.* (ii) in Art. 7. The series ρ cannot be applied there by choosing $a = 1$. We shall see that the series Σ is applicable.

Choose $a = 1$, then

$$x = 6, \qquad \phi'(a) = -3, \qquad \phi''(a) = 28,$$

$$\phi'''(a) = 48, \qquad \phi^{iv}(a) = 24.$$

Now

$$\left\{ \frac{2x}{\phi''(a)} \right\}^{\frac{1}{2}} = \left(\frac{12}{28} \right)^{\frac{1}{2}} = \cdot 65466,$$

$$\frac{\phi'(a)}{\phi''(a)} = \frac{-3}{28} = -\cdot 10714,$$

$$\frac{1}{3} \frac{x\phi'''(a)}{\{\phi''(a)\}^2} = \frac{2 \times 48}{28 \times 28} = \cdot 12245.$$

Hence

$$a_2 = 1 \pm \cdot 65466 + \cdot 10714 - \cdot 12245,$$

$$= \cdot 98469 \pm \cdot 65466,$$

$$= 1 \cdot 63935 \text{ or } \cdot 33003,$$

Calculating also $a_3\xi^3$ we find $a_3\xi^3 = \mp \cdot 004116$.

Therefore $\qquad a_3 = 1 \cdot 63935 - \cdot 004116,$

or $\qquad a_3 = \cdot 33003 + \cdot 004116.$

Thus the pair of roots correct to the second decimal figure is 1·63, ·33. If greater accuracy be desired instead of proceeding further along the series Σ it is advisable to choose for a each of the values 1·63, ·33 separately and to apply the series ρ.

Case when the roots are close together

11. In the application of the series Σ to find a pair of nearly equal (real) roots it is necessary to choose a such that $|\phi'(a)|$ is small and x and $\phi''(a)$ have the same sign. When the roots of the pair are close together the choice of a is difficult to be guessed, but we are guided to the required one by means of the root of $\phi'(z)=0$

which lies between the pair. The principle will be illustrated in the following example.

Ex. (*iii*). Solve $z^4+8z^3-70z^2-144z+936=0.$

Let us consider the pair of roots lying between 4 and 5.

Take $\phi(z)=z^4+8z^3-70z^2-144z$;—

then $\phi'(z)=4z^3+24z^2-140z-144,$

$\phi''(z)=12z^2+48z-140,$

$\phi'''(z)=24z+48,$

$\phi^{iv}(z)=24.$

Choose $a=4$, then $\phi(a)=-928,\quad \phi'(a)=-64,$

$\phi''(a)=244,\qquad \phi'''(a)=144,\qquad \phi^{iv}(a)=24;$

also $\iota=-936-\phi(a)=-936+928=-8.$

The signs of x and $\phi''(a)$ are not the same.

We proceed to find the root of $\phi'(z)=0$ lying between the pair. Applying the series ρ, the root is given by

$$4+\frac{64}{244}-\frac{64^2}{\underline{|2}}\cdot\frac{144}{244^3}+\frac{64^3}{\underline{|3}}\cdot\frac{3.144^2-244.24}{244^5}-\quad\cdots$$

After simplification we obtain the root as 4·244 correct to three places of decimal.

The value of $|\phi'(a)|$ evidently goes on diminishing as we choose now for a the following values in succession, *viz.*, 4, 4·2, 4·24, 4·244,... Of these the value of a sought must be such that x and $\phi''(a)$ have the same sign. Since $\phi''(a)$ is positive throughout the interval, we are to find now for which of the above values of a, x has a positive sign.

By substitution it is found that for the value 4·24 of a, x is $-\cdot002$, so that we must proceed further. We choose then the next value 4·244 for a. The value of x being ·041 we can now apply the series Σ to get the pair of roots. The labour in computing the values of $\phi(a)$, $\phi'(a)$ when $a=4\cdot244$ cannot be avoided but much of it may be curtailed depending on the degree of approximation required.

When an equation has more than two nearly equal roots in a known interval it is easy to conceive that by varying the mode of collecting the terms in u_A we may find a series applicable in a given case.

12. *Analogous series.* The series ρ and Σ have been obtained from a consideration of *simple collections* alone. But the mode of collecting terms is not necessarily restricted to simple partial collections only. When we consider suitable *multiple collections* also we may, if we please, obtain other series *analogous* to ρ or Σ.

13. Up to the present we have discussed rational and integral equations only but in what follows we shall have occasion to pass on to transcendental equations. It is easy to perceive that the methods we have developed may be applied to transcendental equations as well.

Equivalent series. The equation $\phi(z)=\phi(a)+x$ may be put in a form $\phi(z)=\phi(a)\cdot k$ where k stands for the quantity $1+\frac{x}{\phi(a)}$. By taking logarithms of both sides the equation $\phi(z)=\phi(a)\cdot k$ is expressible in the standard form $\log\phi(z)=\log\phi(a)+\log k$, when $\phi(z)=\phi(a)+x$ and $\log\phi(z)=\log\phi(a)+\log k$ are said to be *equivalent equations*. Evidently a succession of *equivalent equations* may be formed. To each of these equivalent equations the series ρ or Σ (as the case may be) being applicable a succession of *equivalent series* may be obtained.

Transformation of Equations

14. In applying the series ρ or Σ to a particular equation some preliminary transformation may appear advantageous. There are only two kinds of transformations, which may be called *rooted* and *non-rooted*. Those transformations which retain the roots in the transformed equation same as those in the original are said to be *rooted*; otherwise the transformations are *non-rooted*. In non-rooted transformations we are given the relations which connect the roots of the transformed equation with those of the original.

15. To understand the nature of the advantage gained by a transformation we take the equation in *Ex.* (*ii*, *b*), Art 7. When $a=2$, the series ${}^{-\phi}\rho_{\phi}$ does not yield the root as it is not sufficiently convergent. Let us consider now the following transformation:

Divide the equation $z^4+4z^3-4z^2-11z+4=0$ throughout by z^2. This transformation is a rooted one and the resulting equation is

$$z^2+4z-4-11z^{-1}+4z^{-2}=0.$$

Here $$\phi(z)=z^2+4z-4-11z^{-1}+4z^{-2},$$

$$\phi'(z)=2z+4+11z^{-2}-8z^{-3},$$

$$\phi''(z)=2-22z^{-3}+24z^{-4},$$

$$\phi'''(z)=66z^{-4}-96z^{-5},$$

$$\phi^{iv}(z)=-264z^{-5}+480z^{-6};$$

and so on.

When $\alpha=2$, $\phi(\alpha)=3{\cdot}5$,

$$\phi'(\alpha)=9{\cdot}75,$$

$$\phi''(\alpha)={\cdot}75,$$

$$\phi'''(\alpha)=1{\cdot}125,$$

$$\phi^{iv}(\alpha)=-{\cdot}75.$$

Substituting the above values in the series $\rho_\phi^{-\phi}$ we have

$$\alpha_3=2-\frac{3{\cdot}5}{9{\cdot}75}-\frac{(3{\cdot}5)^2}{\lfloor 2}\cdot\frac{{\cdot}75}{(9{\cdot}75)^3}-\frac{(3{\cdot}5)^3}{\lfloor 3}\cdot\frac{3({\cdot}75)^2-9{\cdot}75\times1{\cdot}125}{(9{\cdot}75)^5}$$

$$=2-{\cdot}358974-{\cdot}0049562+{\cdot}0007527$$

$$=1{\cdot}6368;$$

which gives the root correct to the third decimal figure.

It is to be noticed that by transformation the new $\phi'(\alpha)$ happens to be large in comparison to $\phi''(\alpha)$.

McClintock's Method

16. Having shown the advantage gained by a transformation we proceed to consider a standard form of an equation which forms the basis of a method by transformation due to E. McClintock.* Given an equation $\phi(z)=0$ (rational and integral) it is possible by means of suitable transformations to express it in McClintock's form $z^n=\omega^n+naf(z)$ where ω is usually an n^{th} root of unity. The series obtained by McClintock for a root of the above equation is deducible from ρ.

* *American Journal of Mathematics*, Vol. XVII, pp. 89-110.

Let us divide the equation $z^n=\omega^n+naf(z)$ throughout by $f(z)$; then $\dfrac{z^n-\omega^n}{f(z)}-na=0$ is a rooted transformed equation. Putting $\psi(z)$ for $\dfrac{z^n-\omega^n}{f(z)}-na$, McClintock's equation is represented by $\psi(z)=0$. Now applying the series ρ we get the one which is identified as the following due to McClintock.

$$\omega+\omega^{1-n}f(\omega)a+\omega^{1-n}\frac{d}{d\omega}\,\omega^{1-n}\{f(\omega)\}^2\,\frac{a^2}{\lfloor 2}$$

$$+\left(\omega^{1-n}\,\frac{d}{d\omega}\,\right)^2\omega^{1-n}\{f(\omega)\}^3\,\frac{a^3}{\lfloor 3}+\qquad\ldots\quad\text{(M)}$$

By means of the above series McClintock explained a method of calculating simultaneously all the roots of an equation. In the course of development he introduced the ideas of 'dominants' and 'spans' in an equation. The recognition of dominants in a given equation is at the root of his method as the series (M) can then be made convergent by certain definite steps. But his attempts seem to have failed in the case of an equation having nearly equal roots. He puts it as follows :—

" That difficulties will arise when we attempt to apply the formula (M) to cases in which there are no obvious dominants is certain. The case of equal roots has already been mentioned as of that nature."

Ascribing this difficulty to the restricted form of the standard equation used by McClintock we propose to consider the following more general form

$$\phi(z)=\phi(a)+xf(z).$$

The equation may be put as

$$\frac{\phi(z)-\phi(a)}{f(z)}=x.$$

Representing

$$\frac{\phi(z)-\phi(a)}{f(z)}$$

by $\psi(z)$ we have the equation expressed in the form $\psi(z)=\psi(a)+x$, where $\psi(a)=0$

Now $$\psi'(a)=\frac{\phi'(a)}{f(a)},$$

$$\psi''(a)=\frac{\phi''(a)}{f(a)}-\frac{2\phi'(a)f'(a)}{\{f(a)\}^2},$$

$$\psi'''(a)=\frac{\phi'''(a)}{f(a)}-\frac{3\phi''(a)f'(a)}{\{f(a)\}^2},$$

$$-3\phi'(a)\frac{f(a)f''(a)-2\{f'(a)\}^2}{\{f(a)\}^3};$$

and so on.

Substituting the above values in the series ${}^{x}\rho_{\psi}$ we can express it in the following convenient form

$$a+x\frac{f(a)}{\phi'(a)}+\frac{x^2}{\underline{|2}}\left(\frac{1}{\phi'(a)}\frac{d}{da}\right)\frac{\{f(a)\}^2}{\phi'(a)}$$

$$+\frac{x^3}{\underline{|3}}\left(\frac{1}{\phi'(a)}\frac{d}{da}\right)^2\frac{\{f(a)\}^3}{\phi'(a)}+\dots$$

when $\psi'(a)$, *i.e.*, $\phi'(a)$ is small or zero we should apply the series Σ to obtain a pair of nearly equal roots. It must be noticed that Mc-Clintock's standard form excludes such a case altogether.

We have not yet considered imaginary roots of an equation. This problem will, however, be taken up in connection with the solution of simultaneous numerical equations which will form the subject of a separate paper in future.

7

WEIERSTRASS *

BY

GANESH PRASAD

1. Karl Theodor Wilhelm Weierstrass was born on the 31st of October, 1815, at Osterfelde in the Münster district of Westphalia. After studying law at the University of Bonn from 1834 to 1838, he went to Münster where he privately studied Mathematics under Gudermann from 1838 to 1840. He was teacher at the Progymnasium of Deutsch-Krone from 1842 to 1848 and head-teacher at the Gymnasium of Braunsberg in East Prussia from 1848 to 1856. He became Honorary Doctor of Philosophy of the University of Königsberg in 1854. In 1856 his research papers on Abelian functions obtained for him an invitation to Berlin as Professor of Pure Mathematics at the Gewerbinstitute and as Member of the Royal Academy of Sciences. In 1856 he also became an extraordinary Professor of the Berlin University and remained there in that capacity until 1864 when he became the third ordinary Professor of Mathematics, the other two being Kummer and Ohm. He remained at the Berlin University until his death which took place after a long illness on the 19th of February, 1897.

2. It is difficult to describe adequately the vast influence which Weierstrass exercised as a lecturer and as a guide of researchers during the forty years of his stay at the Berlin University. Some idea of that influence may be formed by the fact that in the long list of his distinguished pupils are found the names of H. A. Schwarz, Fuchs, Paul du Bois-Reymond, G. Mittag-Leffler, Georg Cantor, Ulissi Dini, Sophie Kowalevsky and Killing. In the circle of his mathematical friends he was looked upon as almost superhuman. According to Professor Mittag-Leffler (*Acta Mathematica*, Vol. 21) it was said: "Weierstrass has indeed something super-human in him. One cannot communicate to him anything which is new to

* Address delivered before the Allahabad University Mathematical Association on the 3rd December, 1924 as a Patron of that Association.

him'; he knows everything." When Professor Mittag-Leffler went to Paris in 1873 to attend the lectures of Hermite, the first words which the great Frenchman addressed to him gave him a shock. "You have made a mistake," said Hermite "you ought to attend the lectures of Weierstrass at Berlin. He is the teacher of all of us." These were sincere words although uttered by such a patriot as Hermite.

3. As regards the *nature* of the work of Weierstrass as a researcher, there can be no doubt that his work brought to a settlement important issues in the theory of functions of real variables and the theory of functions of a complex variable, and placed the theory of elliptic functions and the theory of Abelian functions on simpler bases. Although the main strength of Weierstrass lay in his logical and critical power, in his ability to give strict definitions and to derive rigid deductions therefrom, he was also skilful in the formal treatment of a given question and in deriving for it an algorithm. Using the language of Professor Felix Klein, according to whom, "among mathematicians in general, three main categories may be distinguished," *viz.*, "logicians, formalists and intuitionists," we shall not be wrong in saying that Weierstrass was emphatically a logician and not an intuitionist.

4. That Weierstrass started in his career as a mathematical investigator with a singleness of purpose, is clear from the following remarks made by him when replying to the Presiding Secretary's words of welcome to him on his entering the Academy of Sciences of Berlin as a member. "I ought now to explain in some words what has been up to this time the course of my studies and in what direction I shall direct myself to pursue them. Since the time when under the direction of my highly revered teacher Gudermann, whom I shall always remember with gratitude, I made acquaintance for the first time with the theory of elliptic functions, this comparatively new branch of mathematical analysis has exercised on me a powerful attraction of which the influence on the entire course of my mathematical development has been decisive. This discipline, founded by Euler, cultivated with zeal and success by Legendre but developed in too one-sided a manner, had at that time since a decade undergone a complete transformation because of the introduction of doubly periodic functions by Abel and Jacobi. Those transcendentals, giving to Analysis new quantities of which the properties are remarkable,

find also manifold applications in Geometry and in Mechanics and show thereby that they are the normal fruit of a natural development of Science. But Abel, habituated to place himself always at the most elevated point of view, had found a theorem which, comprehending all the transcendentals resulting from the integration of algebraic differentials, had the same importance for them as Euler's theorem had for the elliptic functions. Cut off in the flower of his age, Abel could not himself pursue his grand discovery, but Jacobi made a second discovery not less important: he demonstrated the existence of periodic functions of several variables of which the principal properties were founded on the theorem of Abel and by which he made known the true significance of that theorem. The actual representation of those quantities of an entirely new kind of which Analysis had not until then an example and the detailed study of their properties became from that time one of the fundamental problems of Mathematics; and as soon as I comprehended the significance and importance of that problem I decided to attempt its solution. It would have been truly foolish if I had thought of solving that problem without having prepared myself by a profound study of the existent means which could aid me and without exercising them on less difficult problems."

5. For the success of his attack on the Abelian functions, Weierstrass planned roughly as follows (see Poincare's paper in *Acta Mathematica*, Vol. 22).

I. To build up the general theory of functions, first that of the functions of one variable and then that of the functions of two variables.

II. The Abelian functions being a natural extension of the elliptic functions, to perfect the theory of these latter transcendentals and to show them in a form in which the generalization becomes clear.

III. To attack lastly the Abelian functions themselves.

6. Although the very first of the total number of sixty papers, published by Weierstrass, was a paper on elliptic functions written in the summer of 1840 and partly published in Vol. 52 of *Crelle's Journal* with the title "On the development of modular functions," there is no doubt that Weierstrass kept the aforementioned plan in view and devoted the succeeding six or seven years to a careful investigation of many important points relating to the theory of functions as is evidenced by the next five of his papers. In the seventh paper, published in the annual report of the Gymnasium

of Braunsberg for the year 1848-1849, he attacked the theory of Abelian Integrals. The next paper, published in 1854 in Vol. 47 of *Crelle's Journal* and entitled "About the theory of Abelian Functions," was followed by a paper on the same subject in 1856 in Vol. 52 of *Crelle's Journal*; these papers brought distinction to Weierstrass and led to his transfer to Berlin. During his stay at the University of Berlin from 1856 onwards he was in the habit of communicating many of his discoveries to his students in his lectures.

7. Weierstrass began to lecture on elliptic functions as early as 1857 but the fundamentally new shape which he gave to the theory of elliptic functions may be said to date from the winter of 1862-1863 when he delivered his first systematic ocurse of lectures on that theory. These lectures he continued for several semesters and the results communicated by him appeared first in the form of H. A. Schwarz's "Formulæ and Theorems for the use of elliptic functions" of which the first edition was begun in 1881 and completed in 1885 and the second edition was completed in 1893. Halphen's famous book of which the last volume appeared in 1891 is based on this book. Weierstrass's theory expounded in his lectures has been given in the 5th Vol. of his "Mathematische Werke" which appeared in 1915.

8. Weierstrass's first lectures on Abelian Functions were delivered in 1863 but it is in the systematic course of lectures which he gave on those functions in the winter semester of 1875-1876 and in the summer semester of 1876 that he developed the subject fully and originally. These lectures first appeared as the fourth volume of Weierstrass's "Mathematische Werke" in 1902.

Weierstrass also published papers on partial differential equations, singularities of algebraic curves, theory of quadratic forms, Projective Geometry, Calculus of Variations and Minimal Surfaces. He was interested in the problem of three bodies and once lectured on synthetic Geometry.

9. Weierstrass's last lectures were delivered in the winter of 1889-1890, and his last paper was communicated to the Royal Academy of Sciences in 1891 and bears the title "New proof of the theorem that every integral rational function of a variable can be represented as a product of linear functions of the same variable."

Soon after this time Weierstrass became seriously ill and never recovered his health.

10. Although Weierstrass's genius was of an order different from that necessary for the production of epoch-making papers on the application of Pure Mathematics to Physics, it will be wrong to think that Weierstrass was not aware of the importance of Applied Mathematics. In his address to the Berlin Academy in 1857, part of which I have already quoted, he emphasized this importance in eloquent language and expressed the hope that more functions would be discovered like Jacobi's theta-function which teaches us into how many perfect squares a given whole number can be broken and how the arc of an ellipse can be measured, and which alone can enable us to express the exact law according to which a pendulum swings.

11. I propose now to attempt a difficult task and that is to place before you, in the short time at my disposal, a few concrete examples of Weierstrass's discoveries; I will speak at length about one of them and simply mention the others :—

(*a*) By giving an example of a function which, while continuous for every value of the variable, did not possess a differential co-efficient for any value of the variable, Weierstrass brought to a settlement an issue which had long agitated the minds of mathematicians. For years before, and after, Ampere's unsuccessful attempt in 1806 to prove that differentiability necessary followed from continuity, most mathematicians believed that, according to what was called the "lex continuitas," such was the case, as they thought that the class of continuous functions was identical with the class of functions representable by graphs. Although Gauss, Dirichlet and Jacobi did not endorse the aforesaid argument, none of them had the conviction that a function which was everywhere continuous but nowhere differentiable could exist. Riemann thought that

$$\sum_{1}^{\infty} \frac{\sin(n^2 x)}{n^2}$$

was such a function but could not prove his statement.

Weierstrass's non-differentiable function

$$\sum_{1}^{\infty} a^n \cos(b^n \pi x),$$

a being any positive proper fraction, b an odd integer subject to the condition that

$$ab > 1 + \frac{3\pi}{2},$$

was communicated by Weierstrass first in one of his lectures in 1861 and then long afterwards in 1872 to the Royal Academy of Sciences of Berlin; it was published by Paul Du Bois Reymond in *Crelle's Journal*, Vol. 79 in 1876. In *Annali di Matematica*, Vol. 8, Dini gave in 1877 a general type of non-differentiable function modelled after Weierstrass. Weierstrass's function was criticized by C. Wiener in *Crelle's Journal*, Vol. 90; to Wiener's criticisms Weierstrass made an effective reply and pointed out Wiener's misunderstanding. In recent years two attempts have been made to deprive Weierstrass of some credit. (1) One was made in 1915 by Dr. Grace Chisolm Young in her Gamble Prize Essay which was published in the *Quarterly Journal of Mathematics*, Vol. 47. Mrs. Young's contention that Cellerier's function,

$$\sum_0^\infty \frac{1}{a^n} \sin (a^n x),$$

a being an even integer, which was alleged by her to have been known to Cellerier before 1861, had nowhere either a progressive or regressive differential co-efficient and was therefore more truly a non-differentiable function than Weierstrass's function which, at an infinite number of points in any interval ever so small, possessed those differential co-efficients of opposite signs although infinitely large, was shown by Mr. Badri Nath Prasad to be wrong. Mr. Badri Nath Prasad proved (See *Proceedings of the Benares Mathematical Society*, Vol. 3, for 1921-1922 and *Jahresbericht der deutschen Mathematiker Vereinigung*, Vol. 31, p. 174) that Cellerier's function was not even non-differentiable as at an infinite number of points in any interval ever so small the function possessed a differential co-efficient infinite in value. (2) The second attempt was made by Dr. M. Jasek of Pilsen (Czecho-Slovakia) in September 1922 before the German Association of Mathematicians when he stated that Bernard Bolzano had given before 1830 an example of a continuous but nowhere differentiable function. It is, however, a matter of some difficulty to accept this statement when it is known that Bolzano writing his book "Para-

doxien des Unendlichen" in 1847-1848, says in the footnote to Art. 37 that a continuous function must be differentiable for every value of the variable with the exception of "isolated values."

(*b*) In his paper, entitled "Definition of analytical functions of a variable by means of algebraic differential equations," which was written in 1842, Weierstrass recognized the possibility of the existence of a function with a natural limit; and the first notice in print of such a limit is to be found in a memoir published by Weierstrass in 1866.

(*c*) Weierstrass's factor theorem, first published in 1876, together with the closely connected Mittag-Leffter's theorem first given in 1877, helped Weierstrass to construct easily the functions $\wp(z)$ and $\sigma(z)$ which enabled him to perfect his theory of elliptic functions.

(*d*) Weierstrass gave a partial differential equation—

$$\frac{\partial^2\sigma}{\partial z^2}=12g_3\frac{\partial\sigma}{\partial g_2}+\tfrac{2}{3}g_2^2\frac{\partial\sigma}{\partial g_3}-\tfrac{1}{12}g_2z^2\sigma$$

for $\sigma(z)$ and used it to expand that function in powers of z up to z^{35}. I have requested Mr. Piare Mohan to expand $\wp(z)$, sn (z), cn(z) and dn (z) and obtain the general terms. I hope he will do his best to complete the investigation on which he has already entered with enthusiasm.

(*e*) Jacobi had given the theorem that a function of n variables could have at the utmost $2n$ periods. Weierstrass gave a new proof of this theorem carefully laying down the conditions under which the theorem is valid. He then studied the properties of those functions of n variables which have $2n$ periods and showed that the properties are analogous to those of elliptic functions.

(*f*) In a paper published in the *Berliner Berichte* in 1866, Weierstrass gave the parametric representation—

$$x=\mathrm{R}\{(1-s^2)\mathrm{F}''(s)+2s\,\mathrm{F}'(s)-2\mathrm{F}(s)\},$$

$$y=\mathrm{R}\{\iota(1+s^2)\mathrm{F}''(s)-2\iota s\,\mathrm{F}'(s)+2\iota\mathrm{F}(s)\},$$

$$z=\mathrm{R}\{2s\mathrm{F}''(s)-2\mathrm{F}'(s)\},$$

for the minimal surface, where s is a complex variable, $\mathrm{F}(s)$ any analytical function of s and R denotes that the real part of the expression within the crooked brackets is to be taken.

(*g*) Weierstrass criticized Dirichlet's principle in 1860 and laid down with care the conditions of its validity.

(*h*) In a paper written as early as 1842, Weierstrass gave a proof of the theorem known in the theory of functions of a complex variable after Laurent who gave a proof in 1843.

Before concluding this address I wish to thank this large audience of young mathematicians who have listened to me with great attention and to express my fervent hope that some of them will feel inspired by what I have said about Weierstrass and take him as their model in their future careers as mathematical researchers.

Review

1

Methoden der Mathematischen Physik—Von R. Courant und D. Hilbert. Erster Band mit 29 Abbildungen. Berlin, Julius Springer, 1924, 450 pages.

This work appears in the series "Die Grundlehren der mathematischen Wissenschaften" as the first volume of the book on the Methods of Mathematical Physics under the joint authorship of Professors Hilbert and Courant of the Göttingen University. Although, in the preface which has been signed by Professor Courant alone, it is stated that the entire responsibility for the contents of the volume is borne by him, he has added the name of Professor Hilbert to his own on the title page because "the scientific and pedagogic tendencies represented here are the children of that direction of mathematical thought which shall always remain connected with the name of Hilbert." Throughout the book the points of view of the calculus of variations play the dominant part, it being the endeavour to characterise mathematical quantities and functions by means of extremal-properties. Nevertheless each chapter forms to a certain extent a self-contained unit and can be, therefore, studied without any knowledge of the rest. Each chapter ends with an article containing supplementary information and problems bearing on the chapter.

The first chapter is headed the algebra of linear transformations and quadratic forms, and deals with linear equations and linear transformations, linear transformations with linear parameter, the "principal-axes-transformation" of a quadratic form, *i.e.*, the transformation of a quadratic form with real co-efficients into a sum of squares, the minimum-maximum property of characteristic values, applications to orthogonal vector systems, Gram's determinant and solution of a system of linear equations corresponding to a form. The supplementary article treats of Hadamard's theorem on the value of a determinant, simultaneous transformation of two quadratic forms in canonical form and the elementary factors of a tensor or a bilinear form.

The second chapter bears the heading, "The problem of the expansions of arbitrary functions in series." At the outset it is stated that the functions under consideration will be understood to be *piecewise smooth*, *i.e.*, to be piecewise continuous and to possess piecewise continuous first derivatives. There are twelve articles which are respectively on orthogonal system of functions, the principle of condensation for functions, independence-measure and dimensions-number, Fourier's series, examples and applications of Fourier's series, Fourier's integral, examples for Fourier's integral, the polynomial of Legendre, the approximation theorem of Weierstrass, examples of other orthogonal systems which include Legendre's, Tchebycheff's, Jacobi's, Hermite's and Laguerre's, the integral equations corresponding to an orthogonal system, and supplements and problems.

The third chapter is on the theory of linear integral equations and contains, after preparatory considerations, the theorems of Fredholm for degenerate kernels, the theorems of Fredholm for an arbitrary kernel, the symmetric kernel and their characteristic values, the development theorem and its applications, the series of Neumann and the reciprocal kernel, the formulæ of Fredholm, the new foundation of the theory, and extension of the limits of the validity of the theory. The supplementary article and problems which conclude the chapter deal, among other things, with singular integral equations, the method of Prof. Erhard Schmidt for the derivation of the theorems of Fredholm, Volterra's integral equations, the method of infinitely many variables and polar integral equations.

The fourth chapter gives the fundamentals of the Calculus of Variations under the headings, the formulation of the problem of the Calculus of Variations, methods of direct solution, the differential equations of the Calculus of Variations, remarks and examples about the integration of Euler's differential equation, boundary conditions, variation-problems with auxiliary conditions, the invariant character of Euler's differential equations, the Green's formulæ, the principle of Hamilton and the differential equations of Physics and a number of problems and supplementary facts in the last article.

The fifth chapter treats of the problems of Mathematical Physics relating to vibrations and characteristic values. The twelve articles of the chapter are headed respectively as follows: general remarks on linear differential equations, vibrations of systems with one degree of freedom, systems with a finite degree of freedom, systems with

an infinite degree of freedom, the non-homogeneous string, the vibrating rod, the vibrating membrane, the vibrating plate, other problems involving characteristic values, the Green's function and the solution of problems involving characteristic values with the help of the theory of integral equations, examples for Green's function, supplements and problems.

The sixth chapter consists of seven articles relating to the application of the Calculus of Variations to the characteristic value problem. The articles are respectively headed the extremal properties of characteristic values, general deductions from the extremal properties of characteristic values, the development-theorem, the asymptotic distribution of characteristic values, the nodal points of characteristic functions, the asymptotic behaviour of Sturm-Liouville's characteristic functions and the extension of the development theorems, supplements and problems.

The seventh chapter deals with special functions defined by means of problems relating to a characteristic value. The five articles of the chapter are respectively on preliminary remarks on linear differential equations of the second order, on the functions of Bessel, on the spherical harmonics of Legendre, on the spherical harmonics of Laplace and on asymptotic expansions.

In recent years Analysis has become more or less free from intuitional points of view; and it has come to pass that many representatives of Analysis have lost the consciousness of the connection of their science with Physics, whilst, on the other hand, to the physicist is frequently missing the understanding for the problems and methods of the mathematician. The aim of Hilbert and Courant's book is to prevent this tendency from gaining ground. And there is not the least doubt that the aim will be realized as soon as the book receives the wide publicity which it deserves.

The book contains much that is new. As an illustration, we may mention the remarkably simple treatment of the problem of the asymptotic distribution of characteristic values. In only twenty pages, most of the results first obtained by Professor Weyl in 1912 and 1913 have been given.

GANESH PRASAD

REVIEW

2

"*The Place of Partial Differential Equations in Mathematical Physics*" by Dr. Ganesh Prasad, M.A., D Sc., Hardinge Professor of Higher Mathematics in the Calcutta University.

The spirit of research may perhaps be described as a spirit of active doubt, and the description would be particularly apt in the sphere of mathematics. We have before us the lectures in published form delivered by Dr. Prasad at the University of Patna in which an attempt is made to foster exactly this spirit of energetic doubt. The subject is the partial differential equations of mathematical physics—a subject to which Dr. Prasad's own contributions are quite substantial. While the subject is placed in a historical setting the author yet gives in a brief compass, glimpses of the course it is likely to take in the future.

For instance, the author has shown how the well-known and well-trusted potential equation of Poisson fails as well as its generalisation by Dr. Petrini for the distribution $\rho=\cos^r \log \frac{1}{r}$ and how also in the conduction of heat and the vibrations of a string the partial differential equations in use cease in certain conditions to have any meaning. Integral equations, on the other hand, are found to offer a better and more precise expression of physical phenomena and the author points to the increasing role integral or integral differential equations are likely to have in future in mathematical physics.

S. C. K.

ON THE CONSTRUCTION OF PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER SATISFYING ASSIGNED CONDITIONS

BY

HARENDRANATH DATTA

(University of Dacca)

At the meeting of the Calcutta Mathematical Society held on August 10, 1924, I read a paper on "A theorem of Lie relating to the theory of Intermediate Integrals of Partial Differential Equations of the Second Order." The theorem* referred to was the following:—

If a partial differential equation of the second order possesses two independent Intermediate Integrals (of the Monge's type), *it can be reduced to the form* $S=0$ *by contact transformations.*

In that paper, it was shown that the possession of two intermediate integrals was a *sufficient condition but not a necessary condition* for equations of the second order which could be transformed into the form by contact transformations. In doing so, I found an example (as, indeed, it was enough to find *at least* one) in the equation† $(1+q^2)r-2pqs+(1+p^2)t=0$ which (*i*) was reducible to the form $S=0$ by a contact transformation, and (*ii*) did not possess any intermediate integral of Monge's type.

This find naturally gave rise to the problem of giving a *practical method* of constructing further examples of this type without much analytical difficulties. The aim of this paper is to supply a method of constructing several equations of the form $ar+bs+t=0$ easily.

In what follows it will be noted that the whole process of construction consists in writing down *special* values of two arbitrary functions and finding out whether the values of a and b thus obtained do *not* satisfy a certain differential equation. And as it is very easy to choose quantities which do not satisfy an equation, the practical difficulty of constructing such equations might be regarded as having been reduced to a minimum.

* See page 295, Art. 254 of Forsyth's *Theory of Differential Equations*, Vol. 6.

† The well-known equation of minimal surfaces.

I

The Method.

(1) A differential equation can be reduced to the form S=0 if its solution can be expressed in the form—

$$x=f(u, v),$$

$$y=g(u, v),$$

$$z=h(u)+k(v).$$

For, in that case, we can transform it in such a way that

$$X=u,\ Y=v \text{ and } Z=z;$$

and this reduces the equation to the form—

$$\frac{\partial^2 Z}{\partial X \partial Y}=0$$

as $Z=h(X)+k(Y)$ is the form of the solution of the transformed equation in terms of X, Y, Z, the new variables.

(2) The transformation used above is a contact transformation* as being a point transformation, it is a special case of contact transformation.

(3) From what we know of the general theory of Intermediate Integrals, it is clear that an equation of the form $f(x, y, z, p, q, r, s, t)=0$ will *not* possess any intermediate integral $u(x, y, z, p, q)=0$ involving an arbitrary function if the subsidiary system of simultaneous equations in the derivatives of u does not possess more than *one* common integral. In the special case of an equation of the form

$$ar+bs+t=0$$

the conditions for the *non-existence* of *two* intermediate integrals of Monge's type can be more definitely stated as follows :—

If the equation $ar+bs+t=0$ does not possess *two* intermediate integrals of Monge's type, and if the roots, σ and ρ, of the quadratic

$$a\mu^2-b\mu+1=0$$

* Lie's definition of the most general contact transformation (see page 315, Art. 128 of Forsyth's *Theory of Differential Equations*, Vol 5) requires that the relation $dZ-PdX-QdY=\rho(dz-pdx-qdy)$ should be identically satisfied, the quantities used having the general meanings given in the book referred to. But here both the sides are *identically* zero.

are unequal, then it is *sufficient* that the equation

$$\begin{vmatrix} \Delta(\lambda)-\Delta''(\sigma), & \Delta(P) \\ \Delta'(\lambda)-\lambda^2, & \Delta'(P)+\Delta''(\rho)-\lambda P \end{vmatrix}=0$$

be satisfied for *at least* one of the two possible assignments of σ and ρ without the vanishing of the four constituents in the determinant,* where

$$\Delta \equiv \frac{\partial}{\partial x} + \sigma\frac{\partial}{\partial y} + (p+\sigma q)\frac{\partial}{\partial z},$$

$$\Delta' \equiv \frac{\partial}{\partial q} - \rho\frac{\partial}{\partial p}$$

$$\Delta'' \equiv (1+q\lambda)\frac{\partial}{\partial z} + \lambda\frac{\partial}{\partial y} + P\frac{\partial}{\partial p},$$

$$\lambda \equiv \frac{\Delta'(\sigma)}{\sigma-\rho} \text{ and } P \equiv \frac{\Delta(\rho)}{\sigma-\rho}.$$

II

The Equation $ar+bs+t=0$.

Consider the case in which a and b are functions of p and q only.

In this case, those equations of the type whose solutions can be expressed by a set of the form

$$x,\ y,\ z = \text{a function of } u + \text{a function of } v$$

are given by

$$u\,v\,r-(u+v)s+t=0,\dagger$$

where $$q-up=f(u)$$

and $$q-vp=g(v),$$

f and g being arbitrary functions of u and v alone respectively.

* See *Ex.* 5 (page 252), *Ex.* 2 (page 258) and Arts. 244-46 of Forsyth's *Theory of Differential Equations*, Vol. 6, where the necessary and sufficient conditions for the possession of one intermediate integral of Monge's type, one intermediate integral involving an arbitrary constant and no intermediate integral are respectively given

† See Prof. Forsyth's paper on "*Partial Differential Equations of the second order having integral systems free from partial quadratures,*" Proc. L. M. S., Vol. 5, page 167.

Hence, by (1) of § I, it is clear that these equations are reducible to the form $S=0$ by contact transformation.

And since a and b are functions of p and q alone, ρ and σ are also functions of p and q alone. Hence,

$$\Delta(\lambda)-\Delta''(\sigma)\equiv 0,$$

$$\Delta(P)\equiv 0,$$

and
$$\Delta'(P)+\Delta''(\rho)-\lambda P\equiv 0;$$

but $\Delta'(\lambda)-\lambda^2$ is not *necessarily* zero.

Hence, in this case, the problem reduces to writing down special values of the arbitrary functions $f(u)$ and $g(v)$ and noting the cases in which $\Delta'(\lambda)-\lambda^2$ is *not* zero.

Examples.

(1) Taking $f(u)=i(1+u^2)^{\frac{1}{2}}$ and $g(v)=i(1+v^2)^{\frac{1}{2}}$ and also

$$\rho=\frac{-pq+i\sqrt{1+p^2+q^2}}{1+q^2} \text{ and } \sigma=\frac{-pq-i\sqrt{1+p^2+q^2}}{1+q^2},$$

we find that

$$\Delta'(\lambda)-\lambda^2\neq 0.$$

Hence, the corresponding equation, *viz.*,

$$(1+q^2)r-2pqs+(1+p^2)t=0$$

gives an example of the type required, a result which was previously* obtained from the first principles.

(2) Again, if we take $f(u)=u-1$ and $g(v)=1-v$, we find that

$$\Delta'(\lambda)-\lambda^2\neq 0.$$

Hence,
$$(q^2-1)r-2(pq-1)s+(p^2-1)t=0$$

is another example of the type required.

In this way, we can construct as many examples of this type as we please.

* See my paper on "*A theorem of Lie relating to the theory of intermediate integrals of partial, etc.*," Bull. Cal. Math. Soc., Vol. XV, No. 2 & 3 (1924-25).

2

On Two Pairs of Factorable Continuants*

BY

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In a recent issue of the Bulletin of the Calcutta Mathematical Society I have discussed one Factorable Continuant.† In the present paper are given four more Factorable Continuants which have been obtained with the help of Heilermann's Theorem.‡ They form two pairs, the continuants of each pair being the numerator and the denominator of the fraction from which they have been deduced. They have all been evaluated determinantally and some algebraic relations, *viz.*, theorems (1), (3), (4) and (5) have been derived.

1. Let $t_{r,n}$ denote the series $q_r + q_{r+1} + q_{r+2} + \ldots\ldots + q_n$

where
$$q_r = \frac{(a^{n+r+1}-1)(a^{n-r+1}-1)}{(a^{2r-1}-1)(a^{2r+1}-1)\,a^{2n-2r+1}}$$

n and r both being positive integers ; $n \geqq r$

Let $t^{(1)}_{r,n}$ $t^{(2)}_{r,n}$ ...... be obtained from $t_{r,n}$ in the following manner :

$$t^{(1)}_{r,n} = q_r t_{r+2,n} + q_{r+1} t_{r+3,n} + \qquad \cdots \quad + q_{n-2} t_{n,n}$$

* For references on the subject see "On the Evaluation of some Factorable Continuants." *Bull. Cal. Math. Soc., Vol. XIII, pp. 71-84.*

† "On a Factorable Continuant," *Bull. Cal. Math. Soc*, *Vol. XIV, pp. 219-238.*

‡ *Journal für Math. 33 (1854), p. 174.* For the general case of the theorem see Haripada Datta, "On the Failure of Heilermann's Theorem." *Proc. Edin. Math. Soc., Vol. XXXV.*

$$t^{(2)}_{r,n} = q_r t^{(1)}_{r+2,n} + q_{r+1} t^{(1)}_{r+3,n} + \dots + q_{n-4}\, t^{(1)}_{n-2,n}$$

$$\dots \qquad \dots \qquad \dots$$

$$t^{(p)}_{r,n} = q_r\, t^{(p-1)}_{r+2,n} + q_{r+1} t^{(p-1)}_{r+3,n} + \dots + q_{n-2p} t^{(p-1)}_{n-2p+2,n}$$

Then $t^{(p)}_{r,n}$

$$= \begin{bmatrix} n-r+2 \\ n-r-2p+1 \end{bmatrix} \Big/ \begin{pmatrix} 2p+2 \\ 2 \end{pmatrix} \begin{pmatrix} 2r+2p-1 \\ 2r-1 \end{pmatrix} a^{(p+1)(2n-2r-2p+1)} \qquad (1)$$

where $\begin{bmatrix} n \\ r \end{bmatrix}$ denotes the product $(a^n-1)(a^{n-1}-1)(a^{n-2}-1)\dots(a^r-1)$

and $\begin{pmatrix} n \\ r \end{pmatrix}$ the product $(a^n-1)(a^{n-2}-1)(a^{n-4}-1)\dots(a^r-1)$.

Proof :—

$$t_{r,n} = \frac{(a^{n-r+2}-1)(a^{n-r+1}-1)}{(a^2-1)(a^{2r-1}-1)a^{2n-2r+1}}$$

for $$q_r = \frac{1}{(a^2-1)\, a^{2n+1}} \left\{ \frac{(a^{n-r+2}-1)(a^{n-r+1}-1)a^{2r}}{a^{2r-1}-1} - \frac{(a^{n-r+1}-1)(a^{n-r}-1)a^{2r+2}}{a^{2r+1}-1} \right\}$$

The difference of the two expressions equivalent to q_{r+1} is obtained by substituting $r+1$ for r in q_r. The other terms may be similarly expressed.

Then theorem (1) may be proved by induction. For

$$q_r t^{(p-1)}_{r+2,n} = \frac{1}{\begin{pmatrix} 2p+2 \\ 2 \end{pmatrix} a^{(p+1)(2n+1)}} \left\{ \frac{\begin{bmatrix} n-p+2 \\ n-r-2p+1 \end{bmatrix} a^{2(p+1)(r+p)}}{\begin{pmatrix} 2r+2p-1 \\ 2r-1 \end{pmatrix}} \dots \right.$$

$$\left.-\frac{\begin{bmatrix} n-r+1 \\ n-r-2p \end{bmatrix}_a a^{2(p+1)(r+p+1)}}{\begin{pmatrix} 2r+2p+1 \\ 2r+1 \end{pmatrix}}\right\} \qquad \ldots \quad (3)$$

assuming that if the theorem holds in the case of $t^{(p-1)}_{r,n}$

Cor.

$$t^{(p)}_{2,n-1} = {}^nS_{n-2p-3} \Big/ {}^nS_{n-1}$$

where ${}^nS_r$ denotes the sum of the products of $1, a, a^2, \ldots a^{n-1}$ taken r at a time.

2. $$\frac{a^{2p-c+1}-1}{a^{2p+1}-1}\,{}^nS_{n-c}\,{}^nS_{n-2p+c-}\;{}^cS_c - \frac{a^{c+1}-1}{a^{2c+1}-1}\,{}^nS_{n-c-1}\,{}^nS_{n-2p+c}\,{}^cS_c$$

$$= -\frac{(a^{n+c+1}-1)(a^{2p-2c}-1)}{(a^{2c+1}-1)(a^{2p+1}-1)}\,{}^nS_c\,{}^nS_{n-2p+c}\;{}^{n-c-1}S_{n-c-1} \qquad \ldots \quad (2)$$

For, after removing the common factors from the left-hand-side expression we have

$$\frac{a^{2p-2c}}{(a^{2p+1}-1)(a^{n-c}-1)} - \frac{1}{(a^{2c+1}-1)(a^{n-2p+c}-1)}$$

$$= -\frac{(a^{n+c+1}-1)(a^{2p-2c}-1)}{(a^{2p+1}-1)(a^{2c+1}-1)(a^{n-c}-1)(a^{n-2p+c}-1)}$$

3. $$\frac{a^{n-p}-1}{a^{2n-2}-1} - \frac{(a^{n-p-2}-1)(a^{2p+1}-1)a^{n-p-1}}{(a^{2n-2}-1)(a^{n+p}-1)} - \frac{a^2-1}{a^{n+p}-1}$$

$$= \frac{(a^{n-p-2}-1)(a^{n-p-3}-1)a^2}{(a^{2n-2}-1)(a^{n+p}-1)}; \qquad \ldots \quad (4)$$

and

$$\frac{a^{n-r}-1}{(a^{n-r-p}-1)(a^{2r+1}-1)(a^p-1)} - \frac{a^p}{(a^p-1)(a^{2r+2p+1}-1)}$$

$$= \frac{a^{n+r+1}-1}{(a^{2r+1}-1)(a^{n-r-p}-1)(a^{2r+2p+1}-1)}. \qquad \ldots \quad (5)$$

These two theorems may be proved easily.

4. The continuant

$$\begin{vmatrix} x-\dfrac{1}{2}\dfrac{a-1}{a^n-1}a^{n-1}, & -x, & & & & \\ 1, & -2, & -x, & & & \\ & 1, & \dfrac{1}{2}\dfrac{(a^n-1)(a^3-1)a^{n-2}}{(a^{n+1}-1)(a^{n-1}-1)}, & -x & & \\ & \cdots & \cdots & \cdots & & \\ & & 1, & (-)^n\dfrac{1}{2}\dfrac{\binom{2n-2}{2}(a^{2n-1}-1)}{\binom{2n-1}{1}}, & -x & \\ & & & & 1, & (-)^n 2 \end{vmatrix}_{2n}$$

$$=(1-x)(a-x)(a^2-x)\ \dots\ (a^{n-1}-x). \qquad \dots \quad (6)$$

Here if e_p denote the element of the principal diagonal in the pth row, then the elements, except the first, of that diagonal are given by

$$e_{4r}=-e_{4r+2}=2; \quad e_{4r+1}=-\frac{1}{2}\frac{\binom{n+2r-1}{n-2r+1}(a^{4r+1}-1)a^{n-2r-1}}{\binom{n+2r}{n-2r}},$$

and

$$e_{4r+3}=\frac{1}{2}\frac{\binom{n+2r}{n-2r}(a^{4r+3}-1)a^{n-2r-2}}{\binom{n+2r+1}{n-2r-1}}.$$

Proof :—

(*i*) On the continuant perform the operation

$$m_{2n}\,\mathrm{col}_{2n}+m_{2n-1}\,\mathrm{col}_{2n-1}+\ \dots\ +m_1\,\mathrm{col}_1$$

where $m_{2n}=1$ and the other multipliers are such as to make all the elements except the first of the last column, vanish. Then from all

the even rows except the last and from all the odd rows except the first, we have two sets of equations, *viz*,

$$m_{2r-1}+(-)^r\,2m_{2r}-xm_{2r+1}=0, \qquad \ldots \quad (7)$$

and

$$m_{2r}+(-)^{r+1}\tfrac{1}{2}k_r m_{2r+1}-xm_{2r+2}=0, \qquad \ldots \quad (8)$$

where

$$k_r=\frac{\binom{n+r-1}{n-r+1}(a^{2r+1}-1)a^{n-r-1}}{\binom{n+r}{n-r}},$$

r being a positive integer which varies from 1 to $n-1$.

From the last row we get

$$m_{2n-1}+(-)^n 2m_{2n}=0. \qquad \ldots \quad (9)$$

From (7) and (9) we have

$$(-)^{r+1}\tfrac{1}{2}m_{2r-1}=m_{2r}-xm_{2r+2}+x^2m_{2r+4}-\ldots$$
$$+(-)^{n-r}x^{n-r}m_{2n}; \qquad \ldots \quad (10)$$

and from (8) and (10) we get

$$m_{2r}=k_r\{m_{2r+2}-xm_{2r+4}+x^2m_{2r+6}-\ldots$$
$$+(-)^{n-r-1}x^{n-r-1}m_{2n}\}+xm_{2r+2}. \qquad \ldots \quad (11)$$

Hence from (10) and (11) we have

$$\tfrac{1}{2}m_{2r-1}=(-)^{r+1}\begin{vmatrix} k_r, & x^2, & & & \\ -1, & k_{r+1}, & x^2 & & \\ \cdots & \cdots & \cdots & & \\ & & -1, & k_{n-2}, & x^2 \\ & & & -1, & k_{n-1} \end{vmatrix}$$

$$\text{and } m_{2r}=\begin{vmatrix} k_r+x, & x^2, & & & \\ -1, & k_{r+1}, & x^2, & & \\ \cdots & \cdots & \cdots & & \\ & & -1, & k_{n-2}, & x^2 \\ & & & -1, & k_{n-1} \end{vmatrix}.$$

Thus the multipliers are all continuants.

(*ii*) From (10) and (11) we also have

$$(-)^{r+1}\tfrac{1}{2}m_{2r-1}=m_{2r}-x(m_{2r+2}-xm_{2r+4}+x^2m_{2r+6}- \quad \dots \quad),$$

$$=k_r m_{2r+2}-(k_r-x)x(m_{2r+4}-xm_{2r+6}+ \quad \dots \quad),$$

$$=(k_{r+1}k_r+x^2)m_{2r+4}-x\{k_{r+1}k_r-x(k_r-x)\}$$

$$\times\{m_{2r+6}-xm_{2r+8}+ \quad \dots \quad \}.$$

In this process of eliminating m_{2r}, m_{2r+2}, etc., the co-efficients are governed by two rules, *viz.*,

$$\left.\begin{aligned} C_p&=k_{r+p-1}C_{p-1}+x^2C_{p-2}, \\ \text{and}\quad D_p&=k_{r+p-1}C_{p-1}-xD_{p-1}, \end{aligned}\right\} \quad \dots (12)$$

where C_p is the co-efficient of m_{2r+2p} and D_p that of the expression

$$-x\{m_{2r+2p+2}-xm_{2r+2p+4}+ \quad \dots \quad \}$$

The two formulæ of (12) may be proved by induction for

$$C_p-D_p=xC_{p-1}\ *$$

Thus eliminating all the multipliers except m_{2n}, we have

$$(-)^{r+1}\tfrac{1}{2}m_{2r-1}=C_{n-r}.$$

(*iii*) Now $C_1=k_r$, $C_2=k_{r+1}k_r+x^2$,

$$C_3=k_{r+2}C_2+x^2C_1=k_{r+2}k_{r+1}k_r\left\{1+\left(\frac{1}{k_rk_{r+1}}+\frac{1}{k_{r+1}k_{r+2}}\right)x^2\right\} \text{ by } (12)$$

$$C_4=k_{r+3}C_3+x^2C_2=k_{r+3}k_{r+2}k_{r+1}k_r$$

$$\times\left\{1+\left(\frac{1}{k_rk_{r+1}}+\frac{1}{k_{r+1}k_{r+2}}+\frac{1}{k_{r+2}k_{r+3}}\right)x^2+\frac{1}{k_rk_{r+1}}\cdot\frac{1}{k_{r+2}k_{r+3}}x^4\right\}$$

* $C_4-D_4=(k_{r+3}C_3+x^2C_2)-(k_{r+3}C_3-xk_{r+2}C_2+x^2k_{r+1}C_1-x^3k_r+x^4)$

$$=x^2C_2+xk_{r+2}C_2-x^2C_2+x^3k_r=xk_{r+2}C_2+x^3k_r=xC_3.$$

The general case may be similarly treated.

$$C_5=k_{r+4}k_{r+3}\dots k_r\left[1+\left(\frac{1}{k_rk_{r+1}}+\frac{1}{k_{r+1}k_{r+2}}\right.\right.$$

$$\left.+\dots\frac{1}{k_{r+3}k_{r+4}}\right)x^2-\left\{\frac{1}{k_rk_{r+1}}\left(\frac{1}{k_{r+2}k_{r+3}}\right.\right.$$

$$\left.\left.\left.+\frac{1}{k_{r+3}k_{r+4}}\right)+\frac{1}{k_{r+1}k_{r+2}}\cdot\frac{1}{k_{r+3}k_{r+4}}\right\}x^4\right]$$

Proceeding in this manner we have

$$C_{n-r}=k_{n-1}k_{n-2}\dots k_{r+1}k_r\left\{1+t_{r+1,\,n-1}x^2+t^{(1)}_{r+1,\,n-1}x^4+\dots\right.$$

$$\left.+t^{(p)}_{r+1,\,n-1}x^{2p+2}+\dots\right\}$$

where the highest power of x is $n-r$ or $n-r-1$ according as $n-r$ is even or odd.

$$\therefore\ m_{2r-1}=(-)^{r+1}\,2\,\frac{\begin{pmatrix}n+r-1\\n-r+1\end{pmatrix}}{\begin{pmatrix}2r-1\\1\end{pmatrix}}{}^{n-r}S_{n-r}$$

$$\times\left\{1+\frac{\begin{bmatrix}n-r\\n-r-1\end{bmatrix}}{(a^2-1)(a^{2r+1}-1)a^{2n-2r-3}}x^2+\dots\right.$$

$$\left.+\frac{\begin{bmatrix}n-r\\n-r-2p-1\end{bmatrix}}{\begin{pmatrix}2p+2\\2\end{pmatrix}\begin{pmatrix}2r+2p+1\\2r+1\end{pmatrix}a^{(p+1)(2n-2r-2p-3)}}x^{2p+2}+\dots\right\}$$

by (1) ... (13)

m_{2r} may be easily obtained with the help of (7) and (13).

$$\text{Hence } m_1=2\left\{\{{}^nS_{n-1}+{}^nS_{n-3}x^2+\dots+{}^nS_{n-2p-3}x^{2p+2}+\dots\right\}$$

$$\text{and } m_2 = {}^{n}S_{n-1} + \frac{a^{n+1}-1}{a^{3}-1}\,{}^{n-1}S_{n-2}\,x + {}^{n}S_{n-3}x^{2} + \dots$$

$$+ {}^{n}S_{n-2p-2}x^{2p+2} + \frac{a^{n+1}-1}{a^{2p+3}-1}\,{}^{n-1}S_{n-2p-3}x^{2p+3} + \quad \dots$$

Thus, after the operation, the first element of the last column is

$$\left(x - \tfrac{1}{2}\,\frac{a-1}{a^{n}-1}\,a^{n-1}\right)m_1 - m_2 x$$

$$= -\left\{(1-x)(a-x)(a^{2}-x)\dots(a^{n-1}-x)\right\}$$

in either case when n is odd or even.

This element of the last column, multiplied by $(-)^{\frac{1}{2}n-1}$ gives the value of the continuant.

5. If we omit x from the first element of the continuant of Art. 4, the value of the continuant is

$$(x+1)(x+a)(x+a^{2}) \;\dots\; (x+a^{n-1}).$$

This may be proved by performing the same operation as given in Art. 4.

Cor. (*i*). From the continuant of Art. 4 when $a=1$, we have

$$\left|\begin{array}{cccccc} x-\frac{1}{2}n, & -x, & & & & \\ 1 & -2, & -x & & & \\ & 1 & \frac{1}{2}\frac{n.3}{(n+1)(n-1)}, & -x & & \\ & \dots & & \dots & & \dots \\ & & & 1, & (-)^{n}\frac{1}{2}\frac{(2n-2,\,2)}{(2n-3,\,1)}, & -x \\ & & & & 1 & (-)^{n}2\,2n \end{array}\right|$$

$$=(1-x)^{n}$$

Here (p, s) denotes the product $\{p(p-2)(p-4)\dots(s+2)s\}$

Cor. (*ii*) If x be omitted from the first element of the principal diagonal in *Cor.* (*i*) the value of the continuant$=(1+x)^{n}$.

6. The continuant

$$\begin{vmatrix} x-\dfrac{a^n-1}{a-1}, & -1 & & & \\ \dfrac{(a^{n+1}-1)(a^{n-1}-1)a}{(a-1)(a^3-1)}, & x, & -1 & & \\ & \dfrac{(a^{n+2}-1)(a^{n-2}-1)a^3}{(a^3-1)(a^5-1)}, & x, & -1 & \\ & \cdots & \cdots & \cdots & \\ & & & \dfrac{(a^{2n-1}-1)(a-1)a^{2n-3}}{(a^{2n-3}-1)(a^{2n-1}-1)}, & x \end{vmatrix}_n$$

$$=\{(x-1)(x-a)(x-a^2)\ldots(x-a^{n-1})\}$$

Proof (*i*). In evaluating this continuant we are to perform n successive operations which may be stated thus :—

If m_r denote the multiplier of the r^{th} column and l that of the last column, then in the k^{th} operation

$$m_r=(-)^{r-1}\frac{\begin{bmatrix} n-1 \\ n-r-k+2\end{bmatrix}\begin{bmatrix} n+r-1 \\ n+r-k+1\end{bmatrix}a^{(r-1)^2}}{\binom{2r-3}{1}\binom{2n-2}{2n-2k+2}\begin{bmatrix} k-1 \\ 1\end{bmatrix}}$$

and $l=\dfrac{1}{x-a^{k-2}}$

where k varies from 1 to n. In the case of the first operation l is, however, governed by the general rule.

(*ii*) We may substitute for the above operations a single operation in which m_r the multiplier of the r^{th} column will be

$$(-)^{r-1}\frac{a^{(r-1)^2}}{\binom{2r-}{1}}\left[\frac{\begin{bmatrix} n-1 \\ 1\end{bmatrix}\begin{bmatrix} n+r-1 \\ 2r\end{bmatrix}}{\begin{bmatrix} 2n-2 \\ 2r\end{bmatrix}\begin{bmatrix} n-r \\ 1\end{bmatrix}}\{(x-1)(x-a)\ldots(x-a^{n-r-1})\}\right.$$

$$\left.+\frac{\begin{bmatrix} n-1 \\ 2\end{bmatrix}\begin{bmatrix} n+r-1 \\ 2r+1\end{bmatrix}}{\binom{2n-2}{2r+2}\begin{bmatrix} n-r-1 \\ 1\end{bmatrix}}\{(x-1)(x-a)\ldots(x-a^{n-r-2})\}+\ldots\right]$$

Thus in m_r the highest power of x is $n-r$ and the co-efficient of x^{n-p-r} is

$$(-)^{r+p-1}\frac{a^{(r-1)^2}}{\begin{pmatrix}2r-3\\1\end{pmatrix}}\cdot\frac{\begin{bmatrix}n-1\\1\end{bmatrix}\begin{bmatrix}n+r-1\\2r+1\end{bmatrix}}{\begin{pmatrix}2n-2\\2r+2\end{pmatrix}\begin{bmatrix}n-r-p\\1\end{bmatrix}}$$

$$\times\left\{\frac{{}^{p}S_{p}}{\begin{bmatrix}p\\1\end{bmatrix}}-\frac{{}^{p-1}S_{p-1}}{\begin{bmatrix}p-1\\1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}}+\frac{{}^{p-2}S_{p-2}}{\begin{bmatrix}p-2\\1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}}\frac{a^{2r+2}-1}{a^{2r+1}-1}\right.$$

$$\left.-\frac{{}^{p-3}S_{p-3}}{\begin{bmatrix}p-3\\1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}}\cdot\frac{a^{2r+4}-1}{a^{2r+1}-1}+\quad\cdots\quad\cdots\right\}$$

Here the last term of the series within the brackets is

$$(-)^{p}\frac{1}{\begin{bmatrix}p\\1\end{bmatrix}}\frac{\begin{pmatrix}2r+2p-2\\2r+p+1\end{pmatrix}}{\begin{pmatrix}2r+p-2\\2r+1\end{pmatrix}},\quad\text{or}\quad(-)^{p}\frac{\begin{pmatrix}2r+2p-2\\2r+p\end{pmatrix}}{\begin{bmatrix}p\\1\end{bmatrix}\begin{pmatrix}2r+p-1\\2r+1\end{pmatrix}}$$

according as p is odd or even.

Now if in the expression within the brackets we put $a=\frac{1}{b}$ and $y=-b^{2r+1}$, then we can show that this expression is zero or

$$(-)^{\frac{p}{2}}b^{p}\frac{(b-1)(b^{3}-1)\ \ldots\ (b^{p-1}-1)}{\{(b^{p}-1)(b^{p-1}-1)\ \ldots\ (b-1)\}\{(1+y)(1+b^{2}y)\ \ldots\ (1+b^{p-2}y)\}}$$

according as p is odd or even.*

Hence in m_r, the co-efficient of x^{n-r-p} is zero or

$$(-)^{r-1}\frac{a^{(r-1)^2+pr+\frac{1}{2}p(p-1)}\begin{bmatrix}n+r-1\\2r+1\end{bmatrix}\begin{bmatrix}n-1\\1\end{bmatrix}}{\begin{pmatrix}2r-3\\1\end{pmatrix}\begin{pmatrix}2n-2\\2r+2\end{pmatrix}\begin{bmatrix}n-r-p\\1\end{bmatrix}\begin{pmatrix}2r+p-1\\2r+1\end{pmatrix}\begin{pmatrix}p\\2\end{pmatrix}}\qquad(15)$$

according as p is odd or even.

* *See* theorem (8) "On a Factorable Continuant," *Bull. Cal. Math. Soc.*, Vol. 14.

(*iii*) If we perform the single operation on the continuant then the first element of the last column is

$$\left(x-\frac{a^n-1}{a-1}\right)m_1-m_2$$

in which the co-efficient of x^{n-p} is

$$(-)^p\ \frac{\begin{bmatrix} n \\ 5 \end{bmatrix}(a^3-1)}{\begin{pmatrix} 2n-2 \\ 6 \end{pmatrix}}\ {}^{n}S_p$$

by (15) in either case when p is odd or even.

Hence

$$\left(x-\frac{a^n-1}{a-1}\right)m_1-m_2=\frac{\begin{bmatrix} n \\ 5 \end{bmatrix}(a^5-1)}{\begin{pmatrix} 2n-2 \\ 6 \end{pmatrix}}$$

$$\times\{(x-1)(x-a)(x-a^2)\ \ldots\ (x-a^{n-1})\} \qquad \ldots \quad (16)$$

(*iv*) The $r+1$th element of the last column is

$$\frac{(a^{n+r}-1)(a^{n-r}-1)a^{2r-1}}{(a^{2r-1}-1)(a^{2r+1}-1)}\ m_r+xm_{r+1}-m_{r+2}$$

in which, if p be even, the co-efficient of x^{n-r-p} is

$$(-)^{r-1}\ \frac{a^{r^2+pr+\frac{1}{2}p(p-1)}\begin{bmatrix} n+r \\ 2r+5 \end{bmatrix}\begin{bmatrix} n-1 \\ 1 \end{bmatrix}}{\begin{pmatrix} 2r-1 \\ 1 \end{pmatrix}\begin{pmatrix} 2n-2 \\ 2r+6 \end{pmatrix}\begin{pmatrix} p-2 \\ 2 \end{pmatrix}\begin{pmatrix} 2r+p-1 \\ 2r+5 \end{pmatrix}\begin{bmatrix} n-r-p-1 \\ 1 \end{bmatrix}}$$

$$\times\left\{\frac{a^{n-r}-1}{(a^{n-r-p}-1)(a^{2r+1}-1)(a^p-1)}\right.$$

$$\left.-\frac{a^p}{(a^p-1)(a^{2r+p+1}-1)}-\frac{a^{n+r+1}-1}{(a^{2r+1}-1)(a^{n-r-p}-1)(a^{2r+p+1}-1)}\right\}=0$$

by (15) and (5).

If p be odd, the co-efficient of x^{n-r-p} in m_r and m_{r+2} and the co-efficient of $x^{n-r-1-p}$ in m_{r+1} are all zero by (15).

Hence in either case when p is odd or even, the element of the last column obtained from $r+1$th row is zero. Thus all the elements, except the first, of the last column, vanish.

(*v*) The multiplier of the last column is

$$(-)^{n-1}\frac{\begin{bmatrix} n-1 \\ 1 \end{bmatrix} a^{(n-1)^2}}{\begin{pmatrix} 2n-3 \\ 1 \end{pmatrix}}$$

and the product of the elements of the lower minor diagonal is

$$\frac{\begin{bmatrix} 2n+1 \\ n+1 \end{bmatrix}\begin{bmatrix} n-1 \\ 1 \end{bmatrix} a^{(n-1)^2}}{\begin{pmatrix} 2n-3 \\ 1 \end{pmatrix}\begin{pmatrix} 2n-1 \\ 3 \end{pmatrix}}$$

From (*iii*), (*iv*) and (*v*) the value of the continuant is readily obtained.

7. If we substitute $x+\frac{a^n-1}{a-1}$ for the first element of the continuant of Art. 6, then the value of the continuant

$$=\{(x+1)(x+a)(x+a^2) \quad \ldots \quad (x+a^{n-1})\}.$$

This may be proved by performing the single operation of Art. 6, on the continuant.

Cor. (*i*) From Art. 6, if $a=1$, we have

$$\begin{vmatrix} x-n, & -1, & & & & \\ \frac{n^2-1^2}{1.3}, & x, & -1, & & & \\ & \frac{n^2-2^2}{1.5}, & x, & -1, & & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \\ & & & & \frac{n^2-(n-1)}{(2n-3)(2n-} & x \end{vmatrix}_n$$

$=(x-1)^n$.

This continuant is due to Mr. Datta.*

Cor. (*ii*) If we substitute $x+n$ for $x-n$ in the first element of the continuant of *Cor,* (*i*) the value of the continuant is

$$(x+1)^n$$

* Haripada Datta, "On the Theory of Continued Fraction," *Proc. Edin. Math. Soc.*, Vol. XXXIV.

3

On the evaluation of a class of Definite Integrals

By

Avadhesh Narayan Singh

(*Calcutta*)

In a review published in March, 1922, in the "*Bulletin of the American Mathematical Society*", vol. 28, pp. 60-61, by Professor Dunham Jackson, the relation

$$\int_0^\infty \frac{\sin^4 t}{t^3}\, dt = \log 2$$

is given with the following remark, "I do not remember seeing a proof of this relation in print; I am personally indebted for various demonstrations of it to Messrs. Grownwall, Landau, M. Reiss and I. Schur."

The object of the present paper is the evaluation of the general class of integrals

$$\int_0^\infty \frac{\sin^m x}{x^n}\, dx.$$

Three methods of arriving at the results have been given by me, and each method has been completely worked out with reference to a particular case of the general integral. The general integral was evaluated by Prof. T. Hayashi* but the method used by him is open to the objection that he obtains his results by proceeding to certain limits. One of the methods used by me is that of contour integration and I believe that this method has never been used by any previous writer for the evaluation of the general integral.

I take this opportunity to express my thanks to Dr. Ganesh Prasad at whose suggestion I took up the work and under whose guidance it was carried out.

* Vide *Nieuw Archief voor wiskunde*, vol. 13, 1921.

Case A: m and n both odd and m≥n

First Method

(1) As an illustration of this method we shall evaluate

$$\int_0^\infty \frac{\sin^5 x}{x^3}\,dx = \int_0^\infty \frac{(e^{ix}-e^{-ix})^5}{(2i)^5 x^3}\,dx$$

$$= \int_0^\infty \frac{(e^{i5x}-e^{-i5x})-5(e^{i3x}-e^{-i3x})+10(e^{ix}-e^{-ix})}{(2i)^5 x^3}\,dx.$$

Let $$f(z)=\frac{e^{i5z}-5e^{i3z}+10e^{iz}}{(2i)^5 z^3}$$

and let us consider

$$\int_C f(z)dz$$

over the contour C formed by

(1) The x-axis from $-R$ to $-r$,

(2) the upper half of the circle $|z|=r$, (γ)

(3) the x-axis from r to R,

(4) the upper half of the circle $|z|=R$, (c)

where $R \longrightarrow \infty$ and $r \longrightarrow 0$.

Now taking the four parts of C separately we have the integral over (1)

$$= \int_{-R}^{-r} f(z)dz = \int_{-R}^{-r} \frac{e^{i5x}-5e^{i3x}+10e^{ix}}{(2i)^5 x^3}\,dx$$

Putting $x=-t$, the above becomes

$$\int_R^r \frac{e^{-i5t}-5e^{-i3t}+10e^{-it}}{(2i)^5 t^3}\,dt$$

$$= -\int_r^R \frac{e^{-i5t}-5e^{-i3t}+10e^{-it}}{(2i)^5 t^3}\,dt.$$

$\therefore$ The sum of the integrals over (1) and (3) is

$$\int_r^R \frac{(e^{i5x}-e^{-i5x})-5(e^{i3x}-e^{-i3x})+10(e^{ix}-e^{-ix})}{(2i)^5\, x^3}dx$$

$$=\int_r^R \frac{\sin^5 x}{x^3}\, dx.$$

Let the integral over (2) taken counter-clockwise be denoted by I. Then

$$\mathrm{I}=\int_\gamma \frac{e^{i5z}-5e^{i3z}+10e^{iz}}{(2i)^5 z^3}\, dz.$$

Considering each member of the numerator separately we have

$$\int_\gamma \frac{e^{i5z}}{z^3}dz=\left[-\frac{e^{i5z}}{2z^2}\right]_\gamma+\left[-\frac{\frac{d}{dz}(e^{i5z})}{2.1.z}\right]_\gamma+\int_\gamma \frac{\frac{d^2}{dz^2}(e^{i5z})}{2.1.z}\,dz,$$

$$\int_\gamma \frac{e^{i3z}}{z^3}dz=\left[-\frac{e^{i3z}}{2z^2}\right]_\gamma+\left[-\frac{\frac{d}{dz}(e^{i3z})}{2.1.z}\right]_\gamma+\int_\gamma \frac{\frac{d^2}{dz^2}(e^{i3z})}{2.1.z}\,dz,$$

$$\int_\gamma \frac{e^{iz}}{z^3}\,dz=\left[-\frac{e^{iz}}{2z^2}\right]_\gamma+\left[-\frac{\frac{d}{dz}(e^{iz})}{2.1.z}\right]_\gamma+\int_\gamma \frac{\frac{d^2}{dz^2}(e^{iz})}{2.1.z}\,dz.$$

$$\therefore\ \mathrm{I}=\left[-\frac{e^{i5z}-5e^{i3z}+10e^{iz}}{(2i)^5 2.z^2}\right]_\gamma$$

$$+\left[-\frac{\frac{d}{dz}(e^{i5z})-5\frac{d}{dz}(e^{i3z})+10\frac{d}{dz}(e^{iz})}{(2i)^5.2.1.z}\right]_\gamma$$

$$+\frac{1}{(2i)^5}\left[\frac{(5i)^2}{2\,!}\int_\gamma \frac{e^{i5z}}{z}dz-\frac{5.(3i)^2}{2\,!}\int_\gamma \frac{e^{i3z}}{z}dz+\frac{10(i)^2}{2\,!}\int_\gamma \frac{e^{iz}}{z}\,dz\right].$$

The first and the second expressions on the right equal

$$\frac{\sin^5 r}{2.r^2} \text{ and } \frac{\frac{d}{dr}(\sin^5 r)}{2.1.r} \text{ respectively.}$$

Both of these tend to zero as $r \longrightarrow 0$.

Also

$$\underset{r \longrightarrow 0}{\mathrm{Lt}} \int_\gamma \frac{e^{ikz}}{z} dz = \pi i,$$

where κ is a positive integer.

$$\therefore \quad \mathrm{I} = \frac{(i)^2 \pi}{(2)^5 i^5 2\,!}(5^2 - 5.3^2 + 10)$$

$$= \frac{(i)^2 \pi}{2^5\ 2\,!}(5^2 - 5\ 3^2 + 10).$$

The integral over (4)

$$= \int_C \frac{e^{i5z} - 5e^{i3z} + 10e^{iz}}{(2i)^5\ z^3} dz.$$

But

$$\int_C \frac{e^{ikz}}{z^3} dz$$

where κ is a positive integer, tends to zero as $\mathrm{R} \longrightarrow \infty$.

$\therefore$ The integral over (4)

$$\int_C \frac{e^{i5z} - 5e^{i3z} + 10e^{iz}}{(2i)^5\ z^3} dz$$

vanishes when $\mathrm{R} \longrightarrow \infty$.

Now, as $f(z)$ is holomorphic within the contour C, by Cauchy's Theorem

$$\int_C f(z) dz,$$

i.e., the sum of the integrals over (1), (2), (3) and (4) is zero.

Therefore we get

$$\int_0^\infty \frac{\sin^5 x}{x^3}\,dx = -\frac{\pi}{2^5\, 2!}(5^2 - 5.3^2 + 10).$$

2. We now integrate

$$\int_0^\infty \frac{\sin^m x}{x^n}\,dx = \int_0^\infty \frac{(e^{ix} - e^{-ix})^m}{(2i)^m x^n}\,dx.$$

In this case we take

$$f(z) = \frac{1}{(2i)^m z^n}\Bigg\{ e^{imz} - me^{i(m-2)z} + \dots \quad \dots$$

$$+(-1)^{\frac{m-1}{2}} \cdot \frac{m!\, e^{iz}}{\{\frac{1}{2}(m-1)\}!\ \{\frac{1}{2}(m+1)\}!} \Bigg\},$$

and integrate over the same contour as in the preceding article.

Reasoning just as in the particular case above we get

(*a*) the sum of the integrals over (1) and (3)

$$= \int_0^\infty \frac{\sin^n x}{x^n}\,dx\,;$$

(*b*) the integral over (2) taken counter-clock-wise, when $r \longrightarrow 0$

$$= \frac{(i)^{m-n}\pi}{2^m\,(n-1)!}\Bigg\{ m^{n-1} - m.(m-2)^{n-1} + \frac{m(m-1)}{2!}(m-4)^{n-1} - \dots$$

$$+(-1)^{\frac{m-1}{2}} \cdot \frac{m!}{\{\frac{1}{2}(m-1)\}!\ \{\frac{1}{2}(m+1)\}!} \Bigg\};$$

(c) the integral over (4) vanishes when $R \longrightarrow \infty$,

$$\therefore \int_0^\infty \frac{\sin^m x}{x^n}\,dx = \frac{(i)^{m-n}.\pi}{2^m\,(n-1)!}\left\{m^{n-1} - \frac{m}{1!}(m-2)^{n-1} + \quad \cdots \right.$$

$$\left. + (-1)^{\frac{m-1}{2}}.\frac{m!}{\{\frac{1}{2}(m-1)\}!\,\{\frac{1}{2}(m+1)\}!}\right\}.$$

Second Method

3.

$$\int_\epsilon^M \frac{\sin^m x}{x^n}\,dx = \left[-\frac{\sin^m x}{(n-1)x^{n-1}}\right]_\epsilon^M + \frac{1}{(n-1)}\int_\epsilon^M \frac{\frac{d}{dx}(\sin^m x)}{x^{n-1}}\,dx$$

$$= \left[-\frac{\sin^m x}{(n-1)x^{n-1}}\right]_\epsilon^M + \left[-\frac{\frac{d}{dx}(\sin^m x)}{(n-1)(n-2)x^{n-2}}\right]_\epsilon^M$$

$$+ \frac{1}{(n-1)(n-2)}\int_\epsilon^M \frac{\frac{d^2}{dx^2}(\sin^m x)}{x^{n-2}}\,dx$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

$$= \left[-\frac{\sin^m x}{(n-1)x^{n-1}}\right]_\epsilon^M + \quad \cdots \quad + \left[-\frac{\frac{d^{n-2}}{dx^{n-2}}(\sin^m x)}{(n-1)!\,x}\right]_\epsilon^M$$

$$+ \frac{1}{(n-1)!}\int_\epsilon^M \frac{\frac{d^{n-1}}{dx^{n-1}}(\sin^m x)}{x}\,dx.$$

When $\epsilon \longrightarrow 0$ and $M \longrightarrow \infty$, we get

$$\int_0^\infty \frac{\sin^m x}{x^n}\,dx = \frac{1}{(n-1)!}\int_0^\infty \frac{\frac{d^{n-1}}{dx^{n-1}}(\sin^m x)}{x}\,dx.$$

Now, when m is odd,

$$\sin^m x = (-)^{\frac{m-1}{2}} \cdot \frac{1}{2^{m-1}} \sum_{r=0}^{\frac{m-1}{2}} (-1)^r \, {}^mC_r \sin(m-2r)x$$

$\therefore \quad \frac{d^{n-1}}{dx^{n-1}} (\sin^m x)$, where n is odd,

$$=(-)^{\frac{m-1}{2}} \frac{1}{2^{m-1}} \sum_{r=0}^{\frac{m-1}{2}} (-1)^{\frac{n-1}{2}} \, {}^mC_r (m-2r)^{n-1} \sin(m-2r)x . (-1)^r$$

$$=\frac{(-1)^{\frac{m+n-2}{2}}}{2^{m-1}} \sum_{r=0}^{\frac{m-1}{2}} (-1)^r \, {}^mC_r (m-2r)^{n-1} \sin(m-2r)x.$$

As
$$\int_0^\infty \frac{\sin px}{x} dx = \frac{\pi}{2},$$

where p is a positive integer,

$$\int_0^\infty \frac{\sin^m x}{x^n} dx = \frac{(-1)^{\frac{m+n-2}{2}} \pi}{2^m (n-1)!} \sum_{r=0}^{\frac{m-1}{2}} (-1)^r \, {}^mC_r (m-2r)^{n-1}$$

Third Method

4. The integral

$$I_n^m = \int_0^\infty \frac{\sin^m x}{x^n} dx$$

(where m and n are odd) can be evaluated by the help of the reduction formula

$$I_n^m = \frac{m(m-1)}{(n-1)(n-2)} I_{n-2}^{m-2} - \frac{m^2}{(n-1)(n-2)} I_{n-2}^m$$

which can be easily proved. The repeated application of the above reduces I_n^m to the sum of a number of integrals of the type I_1^p (where p is odd). The integral I_1^p can be evaluated by expanding the numerator, which will give sines of multiples of x, and dealing with each member separately.

Case B: m and n both even and m≥n

5. This case is quite similar to case A, and any one of the above three methods can be employed. We get the formula

$$\int_0^\infty \frac{\sin^m x}{x^n} = \frac{(i)^{n-m}\,\pi}{2^m(n-1)!} \sum_{r=0}^{\frac{m}{2}-1} (-1)^r\, {}^mC_r (m-2r)^{n-1};$$

(By first method)

or

$$= \frac{(-1)^{\frac{m+n}{2}}\,\pi}{2^m(n-1)!} \sum_{r=0}^{\frac{m}{2}-1} (-1)^r\, {}^mC_r (m-2r)^{n-1}.$$

(By second method)

The application of the third method gives integrals of the form

$$\int_0^\infty \frac{\sin^{2p} x}{x^2}\,dx.$$

All these can be evaluated by expanding the numerator and then "integrating by parts."

Case C: m even and n odd, and m>n>1

First Method

6. Contour integration can be employed for the evaluation of this case, but we have to choose a contour different from the one employed in § 1.

As an illustration we evaluate

$$\int_0^\infty \frac{\sin^4 x}{x^3}\,dx=\int_0^\infty \frac{\cos 4x-4\cos 2x+3}{2^3x^3}\,dx.$$

Take $$f(z) = \frac{e^{4iz}-4e^{2iz}+3}{2^3z^3}$$

and integrate over the contour Γ formed by

(1) the x-axis from r to R,

(2) The part of the circle $|z|=\mathrm{R}$ lying in the positive quadrant, (C)

(3) The y-axis from $i\mathrm{R}$ to ir,

(4) The part of the circle $|z|=r$ lying in the positive quadrant. (C′)

Thus we have

$$\int_\Gamma f(z)dz=\int_r^{\mathrm{R}}\frac{e^{4ix}-4e^{2ix}+3}{2^3x^3}\,dx+\int_{\mathrm{C}}\frac{e^{4iz}-4e^{2iz}+3}{2^3z^3}\,dz$$

$$+\int_{\mathrm{R}}^{r}\frac{e^{-4y}-4e^{-2y}+3}{2^3(i)^3y^3}\,i\,dy+\int_{\mathrm{C}'}\frac{e^{4iz}-4e^{2iz}+3}{2^3z^3}\,dz.$$

Now, the first integral

$$\int_r^{\mathrm{R}}\frac{e^{4ix}-4e^{2ix}+3}{2^3x^3}\,dx$$

$$=\int_r^{\mathrm{R}}\frac{\cos 4x-4\cos 2x+3}{2^3x^3}\,dx+i\int_r^{\mathrm{R}}\frac{\sin 4x-4\sin 2x}{2^3x^3}\,dx$$

$$=\int_r^{\mathrm{R}}\frac{\sin^4 x}{x^3}\,dx+i\int_r^{\mathrm{R}}\frac{\sin 4x-4\sin 2x}{2^3x^3}\,dx. \qquad \dots \quad (a)$$

The second integral

$$\int_{C} \frac{e^{4iz}-4e^{2iz}+3}{2^3z^3}\,dz \quad \text{vanishes when } R \to \infty.$$

The third integral

$$\int_{R}^{r} \frac{e^{-4y}-4e^{-2y}+3}{2^3(i)^3y^3}\,i\,dy = \int_{r}^{R} \frac{e^{-4y}-4e^{-2y}+3}{2^3y^3}\,dy$$

$$= \left[-\frac{e^{-4y}-4e^{-2y}+3}{2^3.\ 2.y^2}\right]_r^R + \left[\frac{4e^{-4y}-8e^{-2y}}{2^3.\ 2.\ y}\right]_r^R$$

$$+\int_{r}^{R} \frac{e^{-4y}-e^{-2y}}{y}$$

$$= \left[-\frac{e^{-4R}-4e^{-2R}+3}{2^4R^2}\right] + \left[\frac{4e^{-4R}-8e^{-2R}}{2^4R}\right]$$

$$+\left[\frac{e^{-4r}-4e^{-2r}+3}{2^4r^2}\right] + \left[-\frac{4e^{-4r}-8e^{-2r}}{2^4r}\right]$$

$$+\int_{r}^{R} \frac{e^{-4y}-e^{-2y}}{y}\,dy. \qquad \ldots \quad (b)$$

The fourth integral

$$\int_{C'} \frac{e^{4iz}-4e^{2iz}+3}{2^3z^3}\,dz$$

$$= \left[-\frac{e^{4iz}-4e^{2iz}+3}{2^3.\ 2.\ z^2}\right]_{ir}^{r} + \left[-\frac{4ie^{4iz}-8ie^{2iz}}{2^3.2.z}\right]_{ir}^{r}$$

$$+\int_{C'} (-16)\frac{e^{4iz}-e^{2iz}}{2^4.\ z}\,dz.$$

$$=\left[-\frac{e^{-4r}-4e^{-2r}+3}{2^4 \cdot r^2}\right]+\left[\frac{4e^{-4r}-8e^{-2r}}{2^4 r}\right]$$

$$+\left[-\frac{e^{4ir}-4e^{2ir}+3}{2^4 r^2}\right]+\left[-\frac{4ie^{4ir}-8ie^{2ir}}{2^4 r}\right]$$

$$-\int_{C'}\frac{e^{4iz}-e^{2iz}}{z}\,dz. \qquad \dots \quad (d)$$

Now, as $f(z)$ is holomorphic within the contour Γ, by Cauchy's Theorem

$$\int_{\Gamma} f(z)\;dz,$$

i.e., the sum of the four integrals vanishes. Therefore taking note of the fact that the real part in the last two brackets out-side the integral sign in (b) cancels with the real part in (d), and then equating the real parts and making $R \longrightarrow \infty$ and $r \longrightarrow 0$, we get

$$\int_0^{\infty}\frac{\sin^4 x}{x^3}\,dx+\int_0^{\infty}\frac{e^{-4y}-e^{-2y}}{y}\,dy=0$$

But by a well known formula* the second of the above two integrals is log 2. Hence

$$\int_0^{\infty}\frac{\sin^4 x}{x^3}\,dx=\log 2.$$

The general case when m is even and n odd can be treated similarly. The formula obtained is

$$\int_0^{\infty}\frac{\sin^m x}{x^n}\,dx=-\frac{(-1)^{\frac{m+n-1}{2}}}{2^{m-1}\,(n-1)!}\sum_{r=0}^{\frac{m-2}{2}}(-1)^r\,{}^mC_r\,(m-2r)^{n-1}\log(m-2r).$$

* See Todhunter's Integral Calculus p. 273.

Second Method

6(a). The method given in § 3, can also be employed for the evaluation of the above case. Before proceeding to the evaluation, we shall establish, an extension of Frullani's theorem

$$\int_0^\infty \frac{\phi(ax)-\phi(bx)}{x}dx=\phi(0)\log\frac{b}{a}.$$

We shall show that

$$\int_0^\infty \left\{A\phi(ax)+B\phi(bx)+C\phi(cx)+\ldots\ldots\right\}\frac{dx}{x},$$

$$=-\{A\log a+B\log b+C\log c+\ldots\}\phi(0),$$

where $A+B+C+\ldots=0$

with the same restrictions on $\phi(x)$ as in the original theorem, (a, b, c...being positive).

Proof:—For convenience let us take four terms

$$\int_0^\infty \{A\phi(ax)+B\phi(bx)+C\phi(cx)+D\phi(dx)\}\frac{dx}{x}$$

$$=\int_0^\infty \Big[\{A\phi(ax)-A\phi(bx)\}+\{(B+A)\phi(bx)-(B+A)\phi(cx)\}$$

$$+\{(A+B+C)\phi(cx)\}-(A+B+C)\phi(dx)\}+(A+B+C+D)\phi(dx)\Big]\frac{1}{x}dx.$$

The last term vanishes, by virtue of the relation assumed between the co-efficients.

Applying Frullani's theorem to the bracketed parts separately we obtain

$-A(\log a-\log b)-(B+A)(\log b-\log c)-(A+B+C)(\log c-\log d)$

$=-\{A\log a+B\log b+C\log c+D\log d\}$.

This establishes the theorem.

7. We now proceed to the evaluation of case C. § 3 gives

$$\int_0^\infty \frac{\sin^m x}{x^n} dx = \frac{1}{(n-1)!} \int_0^\infty \frac{d^{n-1}}{dx^{n-1}} (\sin^m x) \frac{dx}{x},$$

m being even, we have

$$\sin^m x = \frac{(-1)^{\frac{m}{2}}}{2^{m-1}} \sum_{r=0}^{\frac{m}{2}} (-1)^r \, {}^mC_r \cos(m-2r).$$

$$\therefore \quad \frac{d^{n-1}}{dx^{n-1}} (\sin^m x)$$

$$= \frac{(-1)^{\frac{m+n-1}{2}}}{2^{m-1}} \sum_{r=0}^{\frac{m-1}{2}} (-1)^r (m-2r)^{n-1} \, {}^mC_r \cos(m-2r)x.$$

Putting $x=0$ in the above we find that

$$\sum_{r=0}^{\frac{m-2}{2}} (-1)^r \, {}^mC_r (m-2r)^{n-1}$$

where m is even and n odd, $m>n>1$, always vanishes.

Applying the above extension of Frullani's theorem, we get

$$\int_0^\infty \frac{\sin^m x}{x^n} dx = -\frac{(-1)^{\frac{m+n-1}{2}}}{2^{m-1}(n-1)!} \sum_{r=0}^{\frac{m-2}{2}} (-1)^r \, {}^mC_r (m-2r)^{n-1} \log(m-2r).$$

Third Method

8. The reduction formula of § 4 can also be employed to evaluate integrals of the above type. This method is illustrated by the following example.

$$\int_0^\infty \frac{\sin^4 x}{x^3} dx = I_3^4 = \frac{4.3}{2.1} I_1^2 - \frac{4^2}{2.1} I_1^4 ;$$

(*a*) the integral

$$\int_0^\infty \frac{\sin^2 x}{x}\,dx=\int_0^\infty\int_0^\infty \sin^2 x e^{-ax}\,da\,dx;$$

(*b*) the integral

$$\int_0^\infty \frac{\sin^4 x}{x}\,dx=\int_0^\infty\int_0^\infty \sin^4 x e^{-ax}\,da\,dx.$$

Now,

$$\int_0^\infty\int_0^\infty e^{-ax}\sin^2 x\,dx\,da=\int_0^\infty\int_0^\infty \frac{e^{-ax}(1-\cos 2x)}{2}\,dx\,da$$

$$=\operatorname*{Lt}_{\substack{\epsilon\to 0\\ M\to\infty}} \tfrac{1}{2}\int_\epsilon^M\left(\frac{1}{a}-\frac{a}{a^2+2^2}\right)da.$$

And

$$\int_0^\infty\int_0^\infty e^{-ax}\sin^4 x\,dx\,da$$

$$=\frac{1}{2^3}\int_0^\infty\int_0^\infty e^{-ax}(3-4\cos 2x+\cos 4x)\,dx\,da$$

$$=\operatorname*{Lt}_{\substack{\epsilon\to 0\\ M\to\infty}} \frac{1}{2^3}\int_\epsilon^M\left(\frac{3}{a}-4\cdot\frac{a}{a^2+2^2}+\frac{a}{a^2+4^2}\right)da$$

$\therefore$ The principal value of the integral I_2^4 is given by

$$\underset{\substack{\epsilon\to 0\\ M\to\infty}}{\mathrm{Lt}}\left[\,3\int_{\epsilon}^{M}\left(\frac{1}{a}-\frac{a}{a^2+2^2}\right)da-\int_{0}^{M}\left(\frac{3}{a}-\frac{4a}{a^2+2^2}+\frac{a}{a^2+4^2}\right)da\right]$$

$$=\underset{\substack{\epsilon\to 0\\ M\to\infty}}{\mathrm{Lt}}\int_{\epsilon}^{M}\left(\frac{a}{a^2+2^2}-\frac{a}{a^2+4^2}\right)da$$

$$=\underset{\substack{\epsilon\to 0\\ M\to\infty}}{\mathrm{Lt}}\left[\log\frac{(a^2+2^2)^{\frac{1}{2}}}{(a^2+4^2)^{\frac{1}{2}}}\right]_{\epsilon}^{M}$$

$$=\log 2.$$

9. The case when $n=1$ has not been treated. We shall show that

$$\int_0^{\infty}\frac{\sin^{2p}x}{x}\,dx=\infty.$$

We consider the integral

$$\int_0^{n\pi}\frac{\sin^{2p}x}{x}\,dx=\sum_{1}^{n}\int_{(r-1)\pi}^{r\pi}\frac{\sin^{2p}x}{x}\,dx.$$

But

$$\int_{(r-1)\pi}^{r\pi}\frac{\sin^{2p}x}{x}\,dx=\int_0^{\pi}\frac{\sin^{2p}y}{(r-1)\pi+y}\,dy$$

(by putting $x=\{(r-1)\pi+y\}$)

$$\therefore\quad\int_{(r-1)\pi}^{r\pi}\frac{\sin^{2p}x}{x}\,dx>\frac{1}{r\pi}\int_0^{\pi}\sin^{2p}y\,dy.$$

Expanding and integrating the right hand side we get

$$\int_{(r-1)\pi}^{r\pi} \frac{\sin^{2p} x}{x}\,dx > \frac{1}{r\pi}\;\frac{{}^{2p}C_p}{2^{2p-1}}\,\pi$$

$$\therefore\quad \int_0^{n\pi} \frac{\sin^{2p} x}{x}\,dx > \frac{{}^{2p}C_p}{2^{2p-1}} \sum_1^n \frac{1}{r}$$

Hence

$$\int_0^{\infty} \frac{\sin^{2p} x}{x}\,dx = \infty.$$

Case D: m is odd, n is even, m>n

10. This case is similar to case C and the method of contour integration of § 6 or the method of § 7 gives the formula.

$$\int_0^{\infty} \frac{\sin^m x}{x^n}\,dx = -\frac{(-1)^{\frac{m+n-1}{2}}}{2^{m-1}(n-1)!} \sum_0^{\frac{1}{2}(m-1)} (-1)^r\,{}^mC_r\,(m-2r)^{n-1}\log(m-2r).$$

11. A method in all essentials similar to § 8 can also be applied. The method is illustrated by the following example.

$$\int_0^{\infty} \frac{\sin^5 x}{x^2}\,dx = \int_0^{\infty}\int_0^{\infty} \frac{e^{-ax}\sin^5 x}{x}\,da\,dx$$

$$= \frac{1}{2^4}\int_0^{\infty}\int_0^{\infty} \frac{e^{-ax}}{x}(\sin 5x - 5\sin 3x + 10\sin x)\,dx\,da.$$

The principal value of the integral is given by

$$\underset{\substack{\epsilon \longrightarrow 0 \\ M \longrightarrow \infty}}{\mathrm{Lt}} \int_{\epsilon}^{M} \left(\tan^{-1}\frac{5}{a} - 5\tan^{-1}\frac{3}{a} + 10\tan^{-1}\frac{1}{a} \right) da.$$

Putting

$$\frac{a}{\kappa} = \cot\theta \qquad \text{(where } \kappa \text{ is a positive integer)}$$

we get

$$\int_{\epsilon}^{M} \tan^{-1}\frac{\kappa}{a}\, da = -\kappa \int_{a=\epsilon}^{a=M} \theta \operatorname{cosec}^2\theta\, d\theta$$

$$= \kappa \left\{ \Big[-\theta\cot\theta \Big]_{a=\epsilon}^{a=M} + \int_{a=\epsilon}^{a=M} \cot\theta\, d\theta \right\}$$

$$= \kappa \left\{ \Big[-\theta\cot\theta \Big]_{a=\epsilon}^{a=M} + \Big[\log\sin\theta \Big]_{a=\epsilon}^{a=M} \right\}.$$

$$\therefore \underset{\substack{\epsilon \longrightarrow 0 \\ M \longrightarrow \infty}}{\mathrm{Lt}} \int_{\epsilon}^{M} \tan^{-1}\frac{\kappa}{a}\, da = \kappa \left\{ \Big[-\theta\cot\theta \Big]_{\frac{\pi}{2}}^{0} \right.$$

$$\left. + \underset{\substack{\epsilon \longrightarrow 0 \\ M \longrightarrow \infty}}{\mathrm{Lt}} \left[\log\sin\left(\tan^{-1}\frac{\kappa}{a}\right)\right]_{\epsilon}^{M} \right\}$$

$$= -\kappa + \underset{\substack{\epsilon \longrightarrow 0 \\ M \longrightarrow \infty}}{\mathrm{Lt}} \left[\kappa \log\sin\left(\tan^{-1}\frac{\kappa}{a}\right) \right]_{\epsilon}^{M}$$

The principal value of the integral is therefore given by

$$\underset{\substack{\epsilon \to 0 \\ M \to \infty}}{\mathrm{Lt}} \frac{1}{2^4}\left[5 \log \sin\left(\tan^{-1}\frac{5}{a}\right) - 5.3. \log \sin\left(\tan^{-1}\frac{3}{a}\right)\right.$$

$$\left. + 10 \log \sin\left(\tan^{-1}\frac{1}{a}\right)\right]_{\epsilon}^{M}.$$

$$= \underset{M \to \infty}{\mathrm{Lt}} \frac{1}{2^4} \log \left[\frac{\sin^5\left(\tan^{-1}\frac{5}{M}\right)\sin^{10}\left(\tan^{-1}\frac{1}{M}\right)}{\sin^{15}\left(\tan^{-1}\frac{3}{M}\right)}\right]$$

$$= \frac{1}{2^4} \log \frac{5^5 \times (1)^{10}}{3^{15}}$$

$$= \frac{5}{2^4} \log \frac{5}{3^3}.$$

Case E: m and n not restricted to be integers

12. As the different cases coming under this head can be evaluated by the methods dealt with in the previous cases, we shall only give the results, and wherever necessary an indication of the method. In all cases when m is an integer we shall use the formulas

$$\int_0^\infty \frac{\sin bx}{x^a}\, dx = \tfrac{1}{2}\pi b^{a-1}\,\Gamma(a)\,\operatorname{cosec}\left(\tfrac{1}{2}\pi a\right),$$

where $(0<a<2)$; and

$$\int_0^\infty \frac{\cos bx}{x^a}\, dx = \tfrac{1}{2}\pi b^{a-1}\,\Gamma(a)\sec\left(\tfrac{1}{2}\pi a\right),\ (0<a<1),$$

[*Euler*]

The several cases that arise are :—

(*a*) *m is odd, and* $0<n<2$.

Expanding the numerator and considering each member separately we obtain

$$\int_0^\infty \frac{\sin^m x}{x^n}\,dx=\frac{(-1)^{\frac{1}{2}(m-1)}\pi}{2^m\,\Gamma(n)\,\sin\frac{1}{2}n\pi}\sum_{r=0}^{\frac{1}{2}(m-1)}(-1)^r\,{}^mC_r\,(m-2r)^{n-1}$$

(*b*) *m is even, and* $0<n<1$

A process similar to the above shows that

$$\int_0^\infty \frac{\sin^m x}{x^n}\,dx=\infty.$$

(*c*) *m is odd, and* $n=2p+c,\ 0<c<2$ (*where p is an integer*).

We may use any one of the methods § 3 or § 4 The result is

$$\int_0^\infty \frac{\sin^m x}{x^n}\,dx=\frac{(-1)^{\frac{m+2p-1}{2}}\pi}{2^m\,\Gamma(n)\,\sin\frac{1}{2}c\pi}\sum_0^{\frac{1}{2}(m-1)}(-1)^r\,{}^mC_r(m-2r)^{n-1}$$

(*d*) *m is even,* $n=2p+c,\ 0<c<1$ (*where p is a positive integer*).

The result is

$$\int_0^\infty \frac{\sin^m x}{x^n}\,dx=\frac{(-1)^{\frac{m+2p}{2}}\pi}{2^m\,\Gamma(n)\,\cos\frac{1}{2}c\pi}\sum_0^{\frac{m-2}{2}}(-1)^r\,{}^mC_r(m-2r)^{n-1}$$

(*e*) *m is even,* $n=(2p+1)+c,\ 0<c<1$ (*where p is a positive integer*)

$$\int_0^\infty \frac{\sin^m x}{x^n}\,dx=-\frac{(-1)^{\frac{m+2p}{2}}\pi}{2^m\,\Gamma(n)\,\sin\frac{1}{2}c\pi}\sum_0^{\frac{m-2}{2}}(-1)^r\,{}^mC_r(m-2r)^{n-1}$$

(*f*) *m is a fraction whose numerator and denominator are both odd,* $n=1$, $m>n$.

This is a known case *

(*g*) *m is a fraction whose numerator and denominator are both odd, and n is odd,* $m>n+1$.

This case can be evaluated by the help of the reduction formula § 4 or by the method of § 3, and (*f*).

In conclusion we observe that a large class of integrals can be obtained by transformation, *e.g.*, the class of integrals

$$\int_0^\infty \frac{\sin^m \phi(x)}{\{\phi(x)\}^n}\, \phi'(x)dx.$$

* See Whittaker's Modern Analysis, page 262

4

ON THE STABILITY OF VORTEX RINGS OF FINITE CIRCULAR SECTION IN INCOMPRESSIBLE FLUIDS

BY

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In a recent issue of the *Bulletin of the Calcutta Mathematical Society*,* it has been shewn by me that when the vorticity obeys a certain law, it is possible for a ring to move with invariable circular section. The law of vorticity and the velocity of translation were calculated for a ring of a fairly large circular section. The object of the present paper is to consider the stability of that arrangement. I find that the arrangement is quite stable so that even when the circular shape of the cross section is deformed slightly the form of the cross section varies simple-harmonically but the ring continues to move with its velocity unaffected.

Preliminary remarks and definitions

2. Let it be supposed that centroid of the vortex filament lies on the " circular axis " of the ring,

2ω=vorticity, k=strength of the vortex,

c=radius of the circular axis,

ρ, ϕ, z=cylindrical co-ordinates of any point referred to the centre of the circular axis as origin and the axis of the ring as z-axis,

r=distance of any point from the circular axis,

θ=inclination of this distance to the plane of the circular axis, so that $\rho=c-r\cos\theta$,

V=velocity of translation of the ring parrallel to z-axis,

a=radius of the cross section in the undisturbed position,

$$l=\log\frac{8c}{r}-2,\quad s=\frac{r}{c},\quad \lambda=\log\frac{8c}{a}-2,\quad \sigma=\frac{a}{c},$$

* Vol. 14, p. 247-254. 1924.

$$\nabla^2 = \frac{d^2}{dc^2} + \frac{d^2}{dz'^2}, \frac{d}{dc} = \nabla \cos \alpha, \frac{d}{dz'} = \nabla \sin \alpha,$$

ψ=Stokes' stream function,

$$J = \int_0^\pi \frac{c \cos\phi \, d\phi}{[z'^2 + c^2 - 2c\rho'\cos\phi + \rho'^2]^{\frac{1}{2}}}.$$

Then, it can be proved that at any pnoit (ρ', ϕ', z') outside the vortex filament*

$$\psi = \frac{\rho'}{2\pi} \iiint \frac{\omega\rho \cos\phi \, d\phi d\rho dz}{\{(z'-z)^2 + \rho'^2 - 2\rho\rho' \cos\phi + \rho^2\}^{\frac{1}{2}}}$$

$$= \frac{\rho'}{\pi} \iint \omega e^{-x\frac{d}{dc} - z\frac{d}{dz'}} \, dx \, dz \, J \quad \ldots \qquad \ldots \quad (1)$$

where the integral is to be taken over any circular section of the ring.

Now, it has been proved previously † that if

$$\omega = \frac{k}{2\pi a^2}\left[1 + A_2 r^2 \cos 2\theta + A_3 r^3 \cos 3\theta + \ldots\right], \qquad \ldots \quad (2)$$

where
$$A_2 = \frac{36\lambda + 25}{16} \frac{1}{c^2},$$

$$A_3 = \frac{33\lambda + 15}{16} \frac{1}{c^3}, \text{ etc.,} \ldots$$

at any point of the cross section of the ring, then it is possible for the ring to move with invariable circular section.

3. Next, let the central circle of the ring move to a distance z_0 from the plane of xy, and let the cross section in the disturbed position be given by

$$r = a\,[1 + \Sigma\,(\alpha_m \sin m\theta + \beta_m \cos m\theta)], \qquad \ldots \quad (3)$$

* *Bul. Cal. Math. Soc.*, Vol. 13, p. 120.

† *Do.*, Vol. 14, p. 253 result (18).

where α_m, β_m are very small quantities, so that their squares and products etc. can be rejected.

$\therefore$ From (1), (2) & (3),

$$\psi=\frac{k\rho'}{2\pi^2 a^2}\int_0^r\int_0^{2\pi}\left(1+A_2 r^2\cos 2\theta+A_3 r^3\cos 3\theta+\ldots\right)e^{-r\nabla\cos(\theta-a)}\,rdrd\theta\ \mathrm{J},$$

where limit r in given by (3) ... (4)

$$=\frac{k\rho'}{2\pi^2 a^2}\left[\int_0^a\int_0^{2\pi}\left(1+A_2 r^2\cos 2\theta+A_3 r^3\cos 3\theta+\ldots\right)e^{-r\nabla\cos(\theta-a)}\,rdrd\theta\right.$$

$$\left.+\int_0^{2\pi}\int_a^{a[1+\Sigma(\alpha_m\sin m\theta+\beta_m\cos m\theta)]}\left(1+A_2 r^2\cos 2\theta+A_3 r^3\cos 3\theta+\ldots\right)e^{-r\nabla\cos(\theta-a)}\,rdrd\theta\right]\mathrm{J}$$

$$=\frac{k\rho'}{2\pi^2 a^2}\left[\int_0^a\int_0^{2\pi}\left(1+A_2 r^2\cos 2\theta+A_3 r^3\cos 3\theta+\ldots\right)\right.$$

$$\times\{I_0(r\nabla)-2I_1(r\nabla)\cos(\theta-a)+2I_2(r\nabla)\cos 2(\theta-a)\ldots\}rdrd\theta$$

$$+\int_0^{2\pi}\int_0^{a\Sigma(\alpha_m\sin m\theta+\beta_m\cos m\theta)}(1+A_2(a+r)^2\cos 2\theta+A_3(a+r)^3\cos 3\theta+\ldots)$$

$$\left.\times\left\{1-(a+r)\nabla\cos(\theta-a)+\frac{(a+r)^2\nabla^2\cos^2(\theta-a)}{2\,!}\ldots\right\}(a+r)drd\theta\right]\mathrm{J}$$

$$\because\ e^{-r\nabla\cos(\theta-a)}=I_0(r\nabla)-2I_1(r\nabla)\cos(\theta-a)+2I_2(r\nabla)\cos 2(\theta-a)\ \text{etc.}$$

where I_n is Bessel's function of the n^{th} order with imaginary modulus*

$$=\frac{k\rho'}{2\pi^2 a^2}\left[2\pi\left\{\frac{a}{\nabla}I_1(a\nabla)+A_2\cos 2a\frac{a^3}{\nabla}I_3(a\nabla)-A_3\cos 3a\frac{a^4}{\nabla}I_4(a\nabla)+\ldots\right\}\right.$$

* Whittaker—*Modern Analysis* 17·7 and 17·1 *Ex.* 2.

$$+\int_0^{2\pi}\int_0^{a\Sigma(\alpha_m\sin m\theta+\beta_m\cos m\theta)} e^{-a\nabla\cos(\theta-a)}\left\{1-r\nabla\cos(\theta-a)+\frac{r^2\nabla^2\cos^2(\theta-a)}{2!}\ldots\right\}(a+r)drd\theta\Bigg]J$$

$$\left[\because \int_0^a r^{n+1}I_n(r\nabla)\,dr=\frac{a^{n+1}}{\nabla}I_{n+1}(a\nabla)\right]^*$$

$$=\frac{k\rho'}{2\pi^2a^2}\Bigg[\pi a^2\Bigg\{1+\frac{a^2\nabla^2}{8}+\frac{a^4\nabla^4}{192}+\ldots$$

$$+\frac{A_2a^4}{24}\nabla^2\cos2a\left(1+\frac{a^2\nabla^2}{16}+\frac{a^4\nabla^4}{64}+\ldots\right)$$

$$-\frac{a^6A_3}{192}\nabla^3\cos3a\left(1+\frac{a^2\nabla^2}{20}+\ldots\right)+\ldots\ldots\Bigg\}$$

$$+\int_0^{2\pi}a^2\Big\{I_0(a\nabla)-2I_1(a\nabla)\cos(\theta-a)+\ldots+2(-1)^mI_m(a\nabla)\cos m(\theta-a)$$

$$+\ldots\ldots\Big\}\Sigma(\alpha_m\sin m\theta+\beta_m\cos m\theta)d\theta\Bigg]J$$

neglecting α_m^2 etc.

$$=\frac{l\rho'}{2\pi}\Bigg[1+\frac{a^2\nabla^2}{8}+\frac{a^4\nabla^4}{24}+\ldots+\frac{A_2a^4}{24}\nabla^2\cos2a(1+\frac{a^2\nabla^2}{16}+\frac{a^4\nabla^4}{64}+\ldots)$$

$$-\frac{A_3a^3\nabla^3}{192}\cos3a\left((1+\frac{a^2\nabla^2}{20}+\ldots\right)+\ldots$$

$$+2\Sigma(-)^m\left(\alpha_m\sin ma+\beta_m\cos ma\right)I_m(a\nabla)\Bigg]J. \quad\ldots\quad (5)$$

4. Now it can be proved that

$$\nabla^{2n}J=1.\,(-1)(-3)\ldots\ldots(3-2n)\left(\frac{1}{c}\frac{d}{dc}\right)^nJ.$$

* *Modern Analysis*, p. 366, 17·7.

Also $$\frac{d^2}{dc^2}=c^2\left(\frac{1}{c}\frac{d}{dc}\right)^2+\left(\frac{1}{c}\frac{d}{dc}\right),$$

$$\frac{d^3}{dc^3}=c^3\left(\frac{1}{c}\frac{d}{dc}\right)^3+3c\left(\frac{1}{c}\frac{d}{dc}\right)^2 \text{ etc.}$$

Again, $$J\frac{\rho'}{c}=l-\frac{l+1}{2}s\cos\theta+\left(\frac{2l+5}{16}-\frac{l}{16}\cos 2\theta\right)s^2$$

$$+\left(\frac{3l+5}{64}\cos\theta-\frac{3l-1}{192}\cos 3\theta\right)s^3+\left(\frac{12l+11}{2048}\right.$$

$$\left.+\frac{12l+17}{768}\cos 2\theta-\frac{15l-8}{3072}\cos 4\theta\right)s^4+\text{ etc.} \quad \dots \quad \dots \quad (6)$$

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)J=\frac{1}{c^2 s}\left\{-\cos\theta+\left(\frac{2l+3}{4}+\frac{\cos 2\theta}{4}\right)s\right.$$

$$\left.+\left(\frac{4l+1}{32}\cos\theta+\frac{\cos 3\theta}{32}\right)s^2+\dots\right\} \quad \dots \quad \dots \quad (7)$$

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)^2 J=\frac{1}{c^4 s^2}\left\{\cos 2\theta-\frac{\cos\theta+\cos 3\theta}{4}s\right.$$

$$\left.-\left(\frac{12l+9}{32}+\frac{\cos 2\theta}{4}+\frac{\cos 4\theta}{32}\right)s^2-\text{etc.}\right\} \dots \quad \dots \quad (8)$$

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)^3 J=\frac{1}{c^6 s^3}\left\{-2\cos 3\theta+\dots\right\} \quad \dots \quad \dots \quad (9)$$

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)^4 J=\frac{1}{c^8 s^4}\left\{3!\cos 4\theta+\dots\right\} \quad \dots \quad \dots \quad (10)$$

$$\frac{\rho'}{c}\left(\frac{1}{c}\frac{d}{dc}\right)^n J=\frac{1}{c^{2n} s^n}\left\{(-1)^n(n-1)!\cos n\theta+\dots\right.$$

$$\left.\text{terms containing powers of } s\right\} \quad \dots \quad (11)$$

These results are taken from Dyson's paper "Potential of an anchor-ring" Part I and II *Phil. Trans.* A, Vol. 184, 1893 pp. 1086-7.

Hence, from (5)

$$\psi = \frac{kc}{2\pi}\left[l - \frac{l+1}{2}\frac{r}{c}\cos\theta - \frac{a^2}{8cr}\cos\theta \ldots + \sum \frac{a^m}{mr^m}(\alpha_m \sin m\theta + \beta_m \mathrm{Cos}\, m\,\theta)\right]$$

neglecting $\frac{r^2}{c^2}$ and higher powers. ... (12)

Now,

$$\frac{dr}{dt} = \frac{\partial r}{\partial t} + \frac{\partial r}{\partial z_0}\cdot\frac{\partial z_0}{\partial t} + \frac{\partial r}{\partial c}\frac{\partial c}{\partial t} \qquad \ldots \quad (13)$$

$\therefore$ From (3), we have on the surface of the ring,

$$\frac{dr}{dt} = \{1 + \sum(\alpha_m \sin_m m\theta + \beta_m \cos m\theta)\}\dot{a} + a\sum m\dot{\theta}(\alpha_m \cos m\theta - \beta_m \sin m\theta)$$

$$+ (\dot{\alpha}_m \sin m\theta + \dot{\beta}_m \cos m\theta)a\}. \qquad \ldots \quad (14)$$

But

$$\frac{\partial r}{\partial t} = \frac{1}{\rho r}\frac{\partial \psi}{\partial \theta} = \frac{k}{2\pi(c - r\cos\theta}\left\{\frac{l+1}{2}\sin\theta + \frac{1}{8}\frac{a^2}{r^2}\sin\theta\right.$$

$$\left. + \sum \frac{a^m}{r^{m+1}}\alpha_m \cos m\,\theta - \beta_m \sin m\theta)\}\right\}$$

$$= \frac{k}{2\pi c}\left\{\frac{4\lambda+5}{8}\sin\theta + \sum \frac{c}{a}(\alpha_m \cos m\theta - \beta_m \sin m\theta)\right\}$$

approximately on the boundary $r=a$ to a first approximation.

Also

$$r\dot{\theta} = -\frac{1}{\rho}\frac{\partial \psi}{\partial r} = \frac{kc}{2\pi(c - r\cos\theta)}\frac{1}{r} \text{ approximately.}$$

$\therefore \dot{\theta} = \frac{k}{2\pi a^2}$ nearly, when $r=a$.

Also it is easy to see that

$$\frac{\partial r}{\partial c} = \cos\theta, \quad \frac{\partial r}{\partial z_0} = -\sin\theta, \quad \frac{\partial \theta}{\partial c} = -\frac{\sin\theta}{r}, \quad \frac{\partial \theta}{\partial z_0} = -\frac{\cos\theta}{r},$$

Hence from (13) and (14)

$$\frac{k}{2\pi c}\left\{\frac{4\lambda+5}{8}\sin\theta+\sum\frac{c}{a}(a_m\cos m\theta-\beta_m\sin m\theta)\right\}$$

$$-\sin\theta\frac{dz_0}{dt}+\cos\theta.\dot{c}.$$

$$=\dot{a}+\sum\left\{(\dot{a}a_m-\frac{mk\beta_m}{2\pi a}+\dot{a}_m a)\sin m\theta\right.$$

$$+\sum\left((\dot{a}\beta_m+a\dot{\beta}_m+\frac{ma_m k}{2\pi a}\right)\cos m\theta.$$

Substituting the values of

$$\frac{\partial r}{\partial c},\ \frac{\partial r}{\partial z},\ \frac{\partial\theta}{\partial c},\ \frac{\partial\theta}{\partial z_0},\ \text{etc., etc.}$$

whence equating the co-efficients of $\cos m\theta$ and $\sin m\theta$ on both sides, we have,

$$\dot{a}=0,$$

$$\dot{c}=0,$$

$$\dot{z}_0=V=\frac{k}{4\pi c}\left(\lambda+\frac{5}{4}\right)=\frac{k}{4\pi c}\left(\log\frac{8c}{a}-\frac{3}{4}\right).$$

$$\dot{a}a_m+a\dot{a}_m-\frac{km\beta_m}{2\pi a}=-\frac{\beta_m k}{2a\pi},$$

$$\dot{a}\beta_m+a\dot{\beta}_m+\frac{kma_m}{2\pi a}=\frac{a_m k}{2a\pi}.$$

From the last two equations we obtain,

$$\dot{a}_m=-\frac{k\beta_m}{2a\pi}(1-m),$$

$$\dot{\beta}_m=-\frac{ka_m}{2a\pi}(m-1).$$

Hence

$$\ddot{\alpha}_m + \frac{k^2(m-1)^2}{4\pi^2 a^4}\alpha_m = 0,$$

$$\ddot{\beta}_m + \frac{k^2(m-1)^2}{4\pi^2 a^4}\beta_m = 0,$$

which shews that the oscillations are simple harmonic the period being

$$\frac{4\pi^2 a^2}{k(m-1)}.$$

Hence, we find that the motion of the ring is stable.

Thick Ring

5. If the ring be pretty thick, we have from (5)

$$\psi = \frac{k\rho'}{2\pi}\Bigg[1+\left(\frac{a^2}{8}+\frac{A_2 a^4}{24}\right)\frac{1}{c}\frac{d}{dc}+\left\{-\frac{a^4}{192}+A_2 a^4\left(\frac{c^2}{12}+\frac{a^2}{384}\right)\right.$$

$$\left.-\frac{3A_2 a^6 c}{64}\right\}\left(\frac{1d}{cdc}\right)^2+\left\{\frac{a^6}{3072}+\frac{A_2 a^6}{192}\left(c^2-\frac{a^2}{80}\right)\right.$$

$$\left.-\frac{A_2 a^6 c^3}{48}\left(1+\frac{9c^2}{80}\right)\right\}\left(\frac{1d}{cdc}\right)^3+\cdots$$

$$+2\Sigma(-)^m(\alpha_m \sin m\alpha+\beta_m \cos m\alpha)\,I_m(a\nabla)\Bigg]J$$

$$=\frac{kc}{2\pi}\Bigg[l-\frac{l+1}{2}\frac{r}{c}\cos\theta+\left(\frac{2l+5}{16}-\frac{l}{16}\cos 2\theta\right)\frac{r^2}{c^2}$$

$$+\left(\frac{3l+5}{64}\cos\theta-\frac{3l-1}{192}\cos 3\theta\right)\frac{r^3}{c^3}+\ldots$$

$$+\left(\frac{a^2}{8}+\frac{A_2 a^4}{24}\right)\frac{1}{c^2 r^2}\left\{-\cos\theta+\left(\frac{2l+3}{4}+\frac{\cos 2\theta}{4}\right)\frac{r}{c}\right.$$

$$+\left(\frac{4l+1}{32}\cos\theta+\frac{\cos 3\theta}{32}\right)\frac{r^2}{c^2}+\dots\Bigg\}+\left\{\frac{a^6}{3072}+\frac{A_2 a^6}{192}\left(c^2-\frac{a^2}{80}\right)\right.$$

$$\left.-\frac{A_3 a^6 c^3}{48}\left(1+\frac{9\sigma^2}{80}\right)\right\}+\frac{1}{c^6 s^3}\left\{-2\cos 3\theta+\dots\right\}$$

$$+\left\{-\frac{a^4}{192}+{}_2aA^4\left(\frac{c^2}{12}+\frac{a^2}{384}\right)-\frac{3A_2 a^6 c}{64}\right\}\frac{1}{c^4 s^2}$$

$$\times\left\{\cos 2\theta-\frac{\cos\theta+\cos 3\theta}{4}s+\dots\right\}$$

$$+\Sigma\frac{a^m}{mr^m}(\alpha_m\sin m\theta+\beta_m\cos m\theta)\Big].$$

Substituting the values of

$$\frac{1}{c}\frac{dJ}{dc},\quad\left(\frac{1}{c}\frac{d}{dc}\right)^2 J,\ \text{etc.},$$

from (6) to (11), and neglecting s^4 and similar terms,

$$\psi=\frac{kc}{2\pi}\left[l-\frac{l+1}{2}\frac{r}{c}\cos\theta+\left(\frac{2l+5}{16}-\frac{l}{16}\cos 2\theta\right)\frac{r^2}{c^2}\right.$$

$$+\left(\frac{3l+5}{64}\cos\theta-\frac{3l-1}{192}\cos 3\theta\right)\frac{r^3}{c^3}+\frac{\sigma^2}{8}\left\{-\frac{c}{r}\cos\theta\right.$$

$$\left.+\left(\frac{2l+3}{4}+\frac{\cos 2\theta}{4}\right)+\left(\frac{4l+1}{32}\cos\theta+\frac{\cos 3\theta}{32}\right)\frac{r}{c}\right\}$$

$$-\frac{36\lambda+25}{384}\frac{\sigma^4 c}{r}\cos\theta+\left\{-\frac{1}{192}+\frac{36\lambda+25}{192}\right\}\frac{\sigma^4}{s^2}$$

$$\times\left\{\cos 2\theta-\frac{\cos\theta+\cos 3\theta}{4}s+\dots\right\}+\left\{\frac{1}{3072}+\frac{36\lambda+25}{16\times 192}\right.$$

$$\left.-\frac{33\lambda+15}{768}\right\}\frac{\sigma^6}{s^3}\left(-2\cos 3\theta+\dots\ \dots\right)$$

$$\left.+\Sigma\frac{a^m}{mr^m}\left(\alpha_m\sin m\theta+\beta_m\cos m\theta\right)\right]$$

$$= \frac{kc}{2\pi}\left[l - \frac{l+1}{2}\frac{r}{c}\cos\theta + \left(\frac{2l+5}{16} - \frac{l}{16}\cos 2\theta\right)\frac{r^2}{c^2}\right.$$

$$+\left(\frac{3l+5}{64}\cos\theta - \frac{3l-1}{192}\cos 3\theta\right)\frac{r^3}{c^3} + \frac{\sigma^2}{8}\left\{-\frac{c}{r}\cos\theta\right.$$

$$\left.+\left(\frac{2l+3}{4} + \frac{\cos 2\theta}{4}\right) + \left(\frac{4l+1}{32}\cos\theta + \frac{\cos 3\theta}{32}\right)\frac{r}{c}\right\}$$

$$-\frac{36\lambda+25}{384}\frac{\sigma^4 c}{r}\cos\theta + \frac{3\lambda+2}{16}\frac{\sigma^4}{s^2}\left(\cos 2\theta - \frac{\cos\theta + \cos 3\theta}{4}s\right)$$

$$\left.+\frac{48\lambda+17}{768}\frac{\sigma^6}{s^3}\cos 3\theta + \Sigma\frac{a^m}{mr^m}\left(a_m \sin m\theta + \beta_m \cos m\theta\right)\right].$$

$$\frac{\partial r}{\partial t} = \frac{1}{\rho r}\frac{\partial\psi}{\partial\theta} = \frac{kc}{2\pi r(c - r\cos\theta)}\left[\frac{l+1}{2}\frac{r}{c}\sin\theta + \frac{l}{8}\frac{r^2}{c^2}\sin 2\theta\right.$$

$$-\left(\frac{3l+5}{64}\sin\theta - \frac{3l-1}{64}\sin 3\theta\right)\frac{r^3}{c^3} + \frac{\sigma^2}{8}\left\{\frac{c}{r}\sin\theta - \frac{\sin 2\theta}{2}\right.$$

$$\left.-\left(\frac{4l+1}{32}\sin\theta + \frac{3\sin 3\theta}{32}\right)\frac{r}{c}\right\} + \frac{36\lambda+25}{384}\frac{\sigma^4 c}{r}\sin\theta$$

$$+\frac{3\lambda+2}{16}\frac{\sigma^4}{s^2}\left(-2\sin 2\theta + s\,\frac{\sin\theta + 3\sin 3\theta}{4}\right)$$

$$\left.-\frac{48\lambda+17}{256}\frac{\sigma^6}{s^3}\sin 3\theta + \Sigma\frac{a^m}{r^m}\left(a_m \cos m\theta - \beta_m \sin m\theta\right)\right]$$

On the surface of the ring from (3)

$$\frac{\partial r}{\partial t} = \frac{k}{2\pi c}\left[1 + \sigma\cos\theta + \sigma^2\cos^2\theta + \quad\ldots\right]\left[\frac{\lambda+1}{2}\sin\theta\right.$$

$$-\frac{a_2\sin 2\theta + \beta_2\cos 2\theta}{2}\sin\theta + \frac{\lambda}{8}\sigma\sin 2\theta - \sigma^2\left(\frac{3\lambda+5}{64}\sin\theta\right.$$

$$\left.-\frac{3\lambda-1}{64}\sin 3\theta\right) + \frac{\sigma^2}{8}\left\{\frac{\sin\theta}{\sigma^2}(1 - 2a_2\sin 2\theta + \beta_2\cos 2\theta)\right.$$

$$-\frac{\sin 2\theta}{2}\,\frac{1}{\sigma}-\left(\frac{4\lambda+1}{32}\sin\theta+\frac{3\sin 3\theta}{32}\right)\Bigg\}+\frac{36\lambda+25}{384}\,\sigma^2\sin\theta$$

$$+\frac{3\lambda+2}{16}\,\sigma\left(\,2\sin 2\theta+\sigma\,\frac{\sin\theta+3\sin 3\theta}{4}\right)-\frac{48\lambda+17}{256}\,\sigma^2\sin 3\theta$$

$$+\Sigma\,\frac{1}{\sigma}\left(\alpha_m\cos m\theta-\beta_m\sin m\theta\right)\Bigg].$$

$$=\frac{k}{2\pi c}\left[\frac{4\lambda+5}{8}\left\{\sin\theta+\frac{\sigma\sin 2\theta}{2}+\frac{\sigma^2}{4}(\sin 3\theta+\sin\theta)\right\}\right.$$

$$-\frac{4\lambda+5}{16}\,\sigma\left\{\sin 2\theta+\sigma\,\frac{\sin\theta+\sin 3\theta}{2}\right\}-\frac{3}{8}\Bigg\{\alpha_2(\cos\theta-\cos 3\theta)$$

$$+\beta_2(\sin 3\theta-\sin\theta)\Bigg\}+\frac{60\lambda+11}{768}\,\sigma^2\sin\theta+\frac{\alpha_2}{2}\,(\cos\theta+\cos 3\theta)$$

$$\left.-\frac{\beta_2}{2}\,(\sin 3\theta+\sin\theta)+\Sigma\,\frac{1}{\sigma}\,(\alpha_m\cos m\theta-\beta_m\sin m\theta)\right]$$

$$=\frac{k}{2\pi c}\left[\sin\theta\left(\frac{4\lambda+5}{8}+\frac{60\lambda+11}{768}\,\sigma^2\right)\right.$$

$$\left.+\Sigma\,\frac{1}{\sigma}\left(\alpha_m\cos m\theta-\beta_m\sin m\theta\right)\right]. \qquad \ldots \quad (15)$$

N.B. In the above, the value of $\dfrac{\partial r}{\partial t}$ has been calculated correct to σ^2; α_m, β_m being supposed to be very small, $\sigma\alpha_m$, $\sigma\beta_m$, etc., have been rejected in comparison with α_m, β_m, etc. Further, it can be proved that α_1, β_1 are of order $\sigma^2\alpha_3$, $\sigma^2\beta_3$ and have, therefore, been left out of consideration; for, the centroid of the vortex filament being supposed to lie on the circular axis of the ring,

$$\int_0^R\int_0^{2\pi}\omega r\cos\theta\, r dr d\theta=0,$$

$$\int_0^R\int_0^{2\pi}\omega r\sin\theta\, r dr d\theta=0,$$

where limit R is given by (3).

From the first equation and from (2) we have after a little simplification,

$$\beta_1 + \tfrac{1}{2}\beta_1 A_2 a^2 + \frac{\beta_3}{2} A_2 a^2 + \frac{\beta_1}{2} A_3 a^3 + \frac{\beta_4 A_3 a^4}{2} + \quad \ldots = 0,$$

which shows that β_1 is at least of the order $A_2 a^2 \beta_3$, *i.e.*, $\sigma^2 \beta_3$ and can therefore, be rejected in comparison with β_3 etc.

Similarly α_1 can be shown to be of the order $\sigma^2 \alpha_3$ and can, therefore, be rejected in comparison with α_3 etc.

Also

$$r\dot{\theta} = -\frac{1}{\rho}\frac{\partial \psi}{\partial r}.$$

Since it is evident from (14) that we are to calculate $a\dot{\theta}$ correct to $\frac{1}{\sigma}$, we have on the surface of the ring (3),

$$a\dot{\theta} = \frac{kc}{2\pi(c - r\cos\theta)}\,\frac{1}{r} = \frac{k}{2\pi c}\left(\frac{1}{\sigma}\right), \text{ approximately} \quad \ldots \quad (16)$$

As before,

$$\frac{\partial r}{\partial c} = \cos\theta, \quad \frac{\partial r}{\partial z_0} = -\sin\theta, \quad \frac{\partial \theta}{\partial c} = -\frac{\sin\theta}{r}, \quad \frac{\partial \theta}{\partial z_0} = -\frac{\cos\theta}{r},$$

Hence from (13), (14), and (15) (16), we have,

$$\dot{a} + \dot{a}\sum_{m=2}(\alpha_m \sin m\theta + \beta_m \cos m\theta) + a\sum m\dot{\theta}\;(\alpha_m \cos m\theta - \beta_m \sin m\theta)$$

$$+ a\sum(\dot{\alpha}_m \sin m\theta + \dot{\beta}_m \cos m\theta)$$

$$= \frac{k}{2\pi c}\left[\sin\theta\left(\frac{4\lambda+5}{8} + \frac{60\lambda+11}{768}\sigma^2\right) + \frac{1}{\sigma}\sum\Big(\alpha_m \cos m\theta\right.$$

$$\left.-\beta_m \sin m\theta\Big)\right] - V\sin\theta + \dot{c}\cos\theta,$$

or

$$\dot{a} + \sum_{m=2}\left\{\sin m\theta\left(\dot{a}\,\alpha_m + a\,\dot{\alpha}_m - \frac{km}{2\pi a}\beta_m\right)\right.$$

$$\left. + \cos m\theta\left(\dot{a}\,\beta_m + a\dot{\beta}_m + \frac{km\alpha_m}{2\pi a}\right)\right\}$$

$$=\frac{k}{2\pi c}\left[\sin\theta\left\{\frac{4\lambda+5}{8}+\frac{60\lambda+11}{768}\sigma^2\right\}+\frac{1}{\sigma}\sum_{m=2}\left(a_m\cos m\theta\right.\right.$$

$$\left.\left.-\beta_m\sin m\theta\right)\right]-\mathrm{V}\sin\theta+\dot{c}\cos\theta.$$

Hence equating the co-efficients of $\sin m\theta$ and $\cos m\theta$ etc., for different values of m, we have

$$\mathrm{V}=\frac{k}{2\pi c}\left(\frac{4\lambda+5}{8}+\frac{60\lambda+11}{768}\sigma^2\right),$$

$$\dot{a}=0,$$

$$\dot{c}=0,$$

$$\dot{a}\,a_m+a\,\dot{a}_m-\frac{km\,\beta_m}{2\pi a}=-\frac{k\,\beta_m}{2a\pi},$$

$$\dot{a}\,\beta_m+a\,\dot{\beta}_m+\frac{kma_m}{2\pi a}=\frac{k\,a_m}{2a\pi},$$

i.e.,

$$\dot{a}_m=+\frac{k\,\beta_m}{2a^2\pi}(m-1),$$

$$\dot{\beta}_m=-\frac{k\,a_m}{2a^2\pi}(m-1),$$

i.e.,

$$\ddot{a}_m+\frac{k^2(m-1)^2}{4a^4\pi^2}a_m=0,$$

which shews that the oscillations are simple harmonic the period being

$$\frac{4\pi^2a^2}{k(m-1)}.$$

Hence we find that the motion of the thick circular vortex ring is stable.

6. In the above we have got the result correct to σ^2 but the method can be readily extended to find out the stability correct to higher powers of σ.

We find that the velocity of translation remains uneffected,* viz

$$\frac{k}{2\pi c}\left\{\frac{4\lambda+5}{8}+\frac{60\lambda+11}{768}\sigma^2\right\}.$$

Hence the ring continues to move with its previous velocity though its form varies simple harmonically.

* See *Bul. Cal. Math. Soc.* Vol. 14, result (15).

5

A NOTE ON THE TIME IN HYPERBOLIC ORBITS

By

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The time of describing any arc in terms of two extreme radii and the chord has been given by Euler in case of parabolic motions of comets and by Lambert in case of elliptic motions of planets in very neat forms. A corresponding result in case of hyperbolic orbits (which are also possible forms of the orbits under the law of universal gravitation) is given in Plummer's book on Dynamical Astronomy, using hyperbolic functions. Below is given a different proof leading to different form of the same result.

Evidently in this case, v denoting the true anomaly of any position P,

$$r^2\frac{dv}{dt}=h=\sqrt{\mu l}=\frac{2\pi a_0^{\frac{3}{2}}}{T_0}\sqrt{l},$$

where a_0 and T_0 are the mean distance and the periodic time of the Earth.

Thus measuring time from the instant of perihelion

$$t=\frac{T_0}{2\pi a_0^{\frac{3}{2}}}\int_0^v\frac{r^2dv}{\sqrt{l}}.$$

Now u denoting the eccentric anomaly of the position P, on the hyperbola

$$r=a(e\sec u-1),\quad \tan v=\frac{b\tan u}{a(e-\sec u)}=\sqrt{\frac{l}{a}}\cdot\frac{\tan u}{e-\sec u}$$

and so

$$dv=\sqrt{\frac{l}{a}}\cdot\frac{\sec u}{e\sec u-1}du.$$

Hence we get

$$t = \frac{T_0}{2\pi}\left(\frac{a}{a_0}\right)^{\frac{3}{2}}\int_0^u (e\sec u - 1)\sec u\, du$$

$$= \frac{T_0}{2\pi}\left(\frac{a}{a_0}\right)^{\frac{3}{2}}\left[e\tan u - \log\tan\left(\frac{\pi}{4}+\frac{u}{2}\right)\right].$$

The time of describing any arc PP′ is therefore given by

$$\tau = \frac{T_0}{2\pi}\left(\frac{a}{a_0}\right)^{\frac{3}{2}}\left[e(\tan u' - \tan u) - \log\left\{\tan\left(\frac{\pi}{4}+\frac{u'}{2}\right)\cot\left(\frac{\pi}{4}+\frac{u}{2}\right)\right\}\right]$$

$$= \frac{T_0}{2\pi}\left(\frac{a}{a_0}\right)^{\frac{3}{2}}\left[2e\sec u\sec u'\sin\frac{u'-u}{2}\cos\frac{u'-u}{2}\right.$$

$$\left. - \log\left\{\sec u\sec u'(1+\sin u')(1-\sin u)\right\}\right]$$

u and u' denoting the eccentric angles for the points P and P′;
and as

$$(1+\sin u')(1-\sin u) = 1 + (\sin u' - \sin u) - \sin u'\sin u$$

$$= 1 + 2\sin\frac{u'-u}{2}\cos\frac{u'+u}{2} - \frac{1}{2}\left\{\cos(u'-u) - \cos(u'+u)\right\}$$

$$= 2\sin\frac{u'-u}{2}\cos\frac{u'+u}{2} + \frac{1}{2}\left\{1-\cos(u'-u)\right\} + \frac{1}{2}\left\{1+\cos(u'+u)\right\}$$

$$= \left(\cos\frac{u'+u}{2} + \sin\frac{u'-u}{2}\right)^2,$$

we get

$$\tau = \frac{T_0}{2\pi}\left(\frac{a}{a_0}\right)^{\frac{3}{2}}\left[2e\sec u\sec u'\sin\frac{u'-u}{2}\cos\frac{u'-u}{2}\right.$$

$$\left. - \log\left\{\sec u\sec u'\left(\cos\frac{u'+u}{2} + \sin\frac{u'-u}{2}\right)^2\right\}\right] \quad \cdots \quad \text{(i)}$$

We also have

$$r=a(e\sec u-1),$$

$$r'=a(e\sec u'-1).$$

So that

$$r+r'=a[e(\sec u'+\sec u)-2]$$

$$=2a\left[e\sec u\sec u'\cos\frac{u'+u}{2}\cos\frac{u'-u}{2}-1\right]. \quad \dots \quad \dots \quad \text{(ii)}$$

Again, κ denoting the chord PP′,

$$\kappa^2=a^2(\sec u'-\sec u)^2+b^2(\tan u'-\tan u)^2$$

$$=4a^2\sec^2 u\sec^2 u'\sin^2\frac{u'-u}{2}\left[\sin^2\frac{u'+u}{2}+(e^2-1)\cos^2\frac{u'-u}{2}\right]$$

$$=4a^2\sec^2 u\sec^2 u'\sin^2\frac{u'-u}{2}\left[e^2\cos^2\frac{u'-u}{2}-\cos u\cos u'\right]. \quad \dots \quad \text{(iii)}$$

Let us substitute

$$e\sqrt{\sec u\sec u'}\cos\frac{u'-u}{2}=\sec\alpha,$$

and

$$\sqrt{\sec u\sec u'}\cos\frac{u'+u}{2}=\sec\beta\,;$$

and as

$$\cos^2\frac{u'+u}{2}-\sin^2\frac{u'-u}{2}=\cos u\cos u',$$

identically, the last assumption gives

$$\sqrt{\sec u\sec u'}\sin\frac{u'-u}{2}=\tan\beta,$$

(i), (ii), (iii) now become respectively

$$\tau=\frac{T_0}{2\pi}\left(\frac{a}{a_0}\right)^{\frac{3}{2}}[\sec\alpha\tan\beta-\log(\sec\beta+\tan\beta)^2], \quad \dots \quad \dots \quad \text{(iv)}$$

$$r+r'=2a[\sec\alpha\sec\beta-1], \quad \dots \quad \dots \quad \dots \quad \dots \quad \text{(v)}$$

$$\kappa=2a\tan\alpha\tan\beta. \quad \dots \quad \dots \quad \dots \quad \dots \quad \text{(vi)}$$

Finally substitute

$$\left.\begin{array}{l}\sec\alpha\sec\beta+\tan\alpha\tan\beta=\sec\eta\\ \sec\alpha\sec\beta-\tan\alpha\tan\beta=\sec\eta'\end{array}\right\} \quad \ldots \quad \ldots \quad \text{(vii)}$$

Then

$$\tan^2\eta=\sec^2\eta-1=(\sec\alpha\sec\beta+\tan\alpha\tan\beta)^2-(\sec^2\alpha-\tan^2\alpha)$$
$$=(\sec\alpha\tan\beta+\sec\beta\tan\alpha)^2$$

gives

$$\left.\begin{array}{r}\tan\eta=\sec\alpha\tan\beta+\sec\beta\tan\alpha\\ \text{and similarly } \tan\eta'=\sec\alpha\tan\beta-\sec\beta\tan\alpha\end{array}\right\} \quad \ldots \quad \ldots \quad \text{(viii)}$$

From (vii) and (viii) we easily derive

$$\tan\eta+\tan\eta'=2\sec\alpha\tan\beta,$$
$$\sec\eta+\tan\eta=(\sec\alpha+\tan\alpha)(\sec\beta+\tan\beta),$$
$$\sec\eta'+\tan\eta'=(\sec\alpha-\tan\alpha)(\sec\beta+\tan\beta),$$

and so $$(\sec\eta+\tan\eta)(\sec\eta'+\tan\eta')=(\sec\beta+\tan\beta).^2$$

Equations (iv) to (vii) now give

$$\left.\begin{array}{l}r+r'+\kappa=2a\,(\sec\eta-1)\\ r+r'-\kappa=2a\,(\sec\eta'-1)\end{array}\right\} \quad \ldots \quad \ldots \quad \text{(ix)}$$

and

$$\tau=\frac{T^\circ}{2\pi}\left(\frac{a}{a_0}\right)^{\frac{3}{2}}\Big[(\tan\eta+\tan\eta')-\log(\sec\eta+\tan\eta)(\sec\eta'+\tan\eta')\Big] \ldots \quad \text{(x)}$$

The auxilliaries η and η' being given in terms of r, r' and κ by (ix), the time of describing the arc is given by (x).

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NOTES AND NEWS

INSTITUT DE MATHÉMATIQUES

de l'Université de Strasbourg

En outre des cours fondamentaux à programmes permanents en vue des examens de licence et d'agrégation, les *cours de recherche* suivants seront professés en 1924-25 et intéresseront particulièrement les candidats au diplôme d'études supérieures et aux Doctorats d'Université ou d'Etat.

Premier Semestre : Novembre 1924 à Février 1925.

M. Bauer : Théorie des quanta (2 h. par semaine).

M. Cerf : Solutions singulières des équations différentielles et aux dérivées partielles (1 h.)

M. Fréchet : Théorie des ensembles abstraits (3 h.)

Second Semestre : Mars 1925-Juin 1925.

M. Bauer : Constitution des atomes (2 h.)

M. Fréchet : Représentation d'une loi empirique par une formule. Adjustement (3 h.)

M. Thiry : Un chapitre de l'hydrodynamique (2 h)

M. Valiron : Théorie nouvelle des fonctions entières et méromorphes (2 h.)

M. Villat : Recherches sur certaines généralisations de l'équation différentielle de Lamé et sur la théorie des surfaces minima (2 h.)

ERRATA

Page	*Line*	*For*	*Read*
139	7	Grownwall	Gronwall
129	7	Reisz	Riesz
141	2	$-e^{i3x}$	$-e^{-i3x}$
148	7	$\left[-\frac{e^{-4}-4e^{-2}+3}{2^4 R^2}\right]$	$\left[-\frac{e^{-4R}4e^{-2R}+3}{2^4 R^2}\right]$
149	7	that the real part, etc.	that the part in the last two brackets outside the integral sign in (*b*) cancels with part in the first two brackets in (*d*)
149	12	$\log 2$	$-\log 2$
151	4	$\cos(m-2r)$	$\cos(m-2r)x$
151	last line	I_1^{n}	I_1^{4}
153	2	$\int_0^M$	$\int_\epsilon^M$
156	last line	see	sec